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A CLASS OF NEW PARALLEL HYBRID ALGEBRAIC MULTILEVEL ITERATIONS*[1)]

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Abstract

For the large sparse system of linear equations with symmetric positive definite block coefficient matrix resulted from suitable finite element discretization of the second-order self-adjoint elliptic boundary value problem, by making use of the algebraic multilevel iteration technique and the blocked preconditioning strategy, we construct preconditioning matrices having parallel computing function for the coefficient matrix and set up a class of parallel hybrid algebraic multilevel iteration methods for solving this kind of system of linear equations. Theoretical analyses show that, besides much suitable for implementing on the high-speed parallel multiprocessor systems, these new methods are optimal-order methods. That is to say, their convergence rates are independent of both the sizes and the levels of the constructed matrix sequence, and their computational workloads are bounded by linear functions in the order number of the considered system of linear equations, respectively.

Key words: Elliptic boundary value problem, System of linear equations, Symmetric positive definite matrix, Multilevel iteration, Parallel method.

1. Introduction

Consider the large sparse system of linear equations

$$Ax = b, \tag{1.1}$$

where, for a fixed positive integer α, $A \in L(R^n)$ is a symmetric positive definite (SPD) matrix, having the blocked form

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1\alpha} \\ \vdots & \ddots & \vdots \\ A_{\alpha 1} & \cdots & A_{\alpha\alpha} \end{pmatrix}, \quad A_{ij} \in L(R^{n_j}, R^{n_i}), \quad i,j = 1,2,\cdots,\alpha; \tag{1.2}$$

$x, b \in R^n$ are the unknown and the known vectors, respectively, having the corresponding blocked forms

$$\begin{cases} x = (x_1^T, x_2^T, \cdots, x_\alpha^T)^T, & x_i \in R^{n_i}, \\ b = (b_1^T, b_2^T, \cdots, b_\alpha^T)^T, & b_i \in R^{n_i}, \end{cases} \quad i = 1,2,\cdots,\alpha; \tag{1.3}$$

$n_i (n_i \le n; i = 1,2,\cdots,\alpha)$ are α given positive integers, satisfying $\sum_{i=1}^{\alpha} n_i = n$. This system of linear equations often arises in suitable finite element discretizations of many second-order

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self-adjoint elliptic boundary value problems. For details, we refer to [3-6]. Therefore, to study efficient numerical methods for getting the approximate solution of (1.1) has both theoretical and applicable meanings.

There has been a vast amount of research literature on iterative methods for solving the system of linear equatons (1.1). Among these methods, the recently developed algebraic multilevel iteration (AMLI) methods (see [3-5, 7, 9-10, 15-17]) are considerably applicable and efficient, because they are optimal-order methods in the sense that their convergence rates are independent of both the sizes and the level numbers of the grids, and their computational workloads are bounded by linear functions about the stepsizes of the finest grids.

To suit the requirement of the parallel high-speed multiprocessor systems, the authors of [6, 8, 18] presented several parallel algebraic multilevel iteration methods for solving the system of linear equations (1.1), which are, substantially, the parallelized variants of the existing AMLI methods originally presented by Axelsson and Vassilevski [3-5] and further studied by Vassilevski [17] and Bai [7], respectively. These parallel algebraic multilevel iteration methods not only inherit the intrinsic advantages of the AMLI methods, but also have nice parallelism. Therefore, they are very suitable for solving the system of linear equations (1.1) on the parallel computing environments.

Through reasonable combinations of the AMLI technique and the block strategy, in this paper we establish a class of new parallel hybrid AMLI methods based upon the abovementioned existing results. Compared with the parallel AMLI methods in [6, 8, 18], these novel ones have less computational complexities. Hence, they can achieve higher parallel computational efficiency. In a careful way, we estimate the relative condition numbers of the new parallel preconditioners and calculate the computational workloads of the resulted parallel hybrid algebraic multilevel preconditioning methods. We demonstrate that the new parallel algebraic multilevel iteration methods are optimal-order methods for both two-dimensional (2-D) and three-dimensional (3-D) problem domains. That is to say, their computational workloads are proportional to the dimension of the linear system (1.1), and their relative condition numbers of the preconditioners are not only independent of the regularity of the solution, but also bounded uniformly with respect to possible jumps of the coefficients of the second-order self-adjoint elliptic boundary value problem as long as these jumps occur only across edges (faces in 3-D) of elements from the coarsest triangulation. At last, we also formulate adaptive procedures for the new parallel hybrid algebraic multilevel iteration methods in order to construct the involved polynomials after each group of fixed recursion steps of the preconditioners. Therefore, these polynomials can vary from one group of the fixed recursion steps to the next.

2. Constructions of the New Methods

Denote $\Lambda = \{1, 2, \cdots, \alpha\}$. For a fixed positive integer l, starting from the blocked matrix $A \in L(R^n)$ in (1.2), we construct a matrix sequence $\{A^{(k)}\}_{k=0}^{l}$ in accordance with the following rule:

$$\begin{cases} A^{(l)} = A, \quad A^{(k)} = \begin{pmatrix} A_{11}^{(k)} & \cdots & A_{1\alpha}^{(k)} \\ \vdots & \ddots & \vdots \\ A_{\alpha 1}^{(k)} & \cdots & A_{\alpha\alpha}^{(k)} \end{pmatrix}, \qquad A_{ij}^{(k)} \in L(R^{n_j^{(k)}}, R^{n_i^{(k)}}), \\ i, j = 1, 2, \cdots, \alpha; \quad k = 0, 1, \cdots, l, \end{cases}$$

where

$$A_{ii}^{(k)} = \begin{pmatrix} C_{ii}^{(k)} & E_{ii}^{(k)} \\ E_{ii}^{(k)^T} & A_{ii}^{(k-1)} \end{pmatrix}, \quad A_{ij}^{(k)} = \begin{pmatrix} C_{ij}^{(k)} & E_{ij}^{(k)} \\ F_{ij}^{(k)} & A_{ij}^{(k-1)} \end{pmatrix}, \qquad i \neq j, \quad i, j \in \Lambda,$$