AN ALGORITHM FOR REDUCING THE MATRIX NORM*

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Abstract

Based on the singular decomposition of 2×2 matrix an algorithm for reducing the matrix norm is presented. Under the optimal choice of the parameters the matrix B after transformation may be considered as "locally normal", that is, the four corresponding elements of BB^T-B^TB are zero.

1. Introduction

Eberlein [1, 2] proposed a Jacobi-like method to compute the eigenvalues and eigenvectors of arbitrary matrix A. At each step the similarity transformation $B=H^{-1}AH$ is needed to reduce the norm of the matrix, and in the real case only 4 of the elements h_{ij} of H differ from those of identity matrix and they are

$$h_{pp}=h_{qq}=\cosh\xi,\ h_{pq}=h_{qp}=\sinh\xi.$$

Later, the same problem was considered, but

$$h_{pp}=h_{qq}=1,\ \ h_{pq}=0,\ h_{qp}=\xi.$$

Since the singular decomposition of any 2×2 non-singular matrix is

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\xi & 0\\ 0 & \eta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

and the third matrix is useless for the norm reduction, hence in this paper we consider

$$h_{pp}=\xi\cos\theta,\ h_{pq}=-\eta\sin\theta,\ h_{qp}=\xi\sin\theta,\ h_{qq}=\eta\cos\theta.$$

In this way, we choose the suitable transformation among all possibilities, not only among the particular family depending on one parameter.

In § 3, excluding the easily-verified particular case where we may directly compute one or two eigenvalue, we prove that there exist the values of parameters θ, ξ and η to minimize the matrix norm. In § 4 we prove that under optimal choice of the parameters, the 4 corresponding elements of BB^T-B^TB are zero. In § 5 the unique problem is discussed. In § § 6—9 we consider how to determine the optimal values in different cases. Generally, the system, which the optimal values satisfy, can be reduced to an algebraic equation of order 8, and the interval including the root needed is located. In a particular case we may get the exact solution.

The speed of the norm reduction and the numerical stability are not to be discussed.

For simplicity, we consider the real matrix only, but obviously it can be generalized to the complex case. Further information will be presented in another paper.

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2. Notations

Let $A=(a_{ij})$ be any arbitrary $N\times N$ real matrix. Let the elements of $N\times N$ real matrix $H=(h_{ij})$ are defined as follows

$$h_{pp}=\xi\cos\frac{\theta}{2},\quad h_{pq}=-\eta\sin\frac{\theta}{2},$$
$$h_{qp}=\xi\sin\frac{\theta}{2},\quad h_{qq}=\eta\cos\frac{\theta}{2},$$
$$p<q\quad\text{fixed},\ \xi>0,\ \eta>0,$$
$$h_{ij}=\delta_{ij},\quad\text{otherwise}.$$

Obviously, $H^{-1}=(h'_{ij})$ exists, and

$$h'_{pp}=\xi^{-1}\cos\frac{\theta}{2},\ h'_{pq}=\xi^{-1}\sin\frac{\theta}{2},$$
$$h'_{qp}=-\eta^{-1}\sin\frac{\theta}{2},\ h'_{qq}=\eta^{-1}\cos\frac{\theta}{2},$$
$$h'_{ij}=\delta_{ij}.$$

Let $H^{-1}AH=B=(b_{ij})$. Obviously, $b_{ij}=a_{ij}$, when $i,\ j\neq p,\ q$.

Let
$$\tau(B)=\sum_j{}' b_{pj}^2+\sum_j{}' b_{qj}^2+\sum_i{}' b_{ip}^2+\sum_i{}' b_{iq}^2+b_{pp}^2+b_{qq}^2+b_{pq}^2+b_{qp}^2,$$

where $\sum'$ denotes that $i,\ j$ run through 1 to N except p and q.

Let
$$a_1=\frac{1}{2}(a_{pq}+a_{qp}),\ a_2=\frac{1}{2}(a_{pq}-a_{qp}),\ a_3=\frac{1}{2}(a_{qq}-a_{pp}),$$
$$a_4=\sum_j{}' a_{pj}^2,\qquad a_5=\sum_j{}' a_{qj}^2,\qquad a_6=\sum_j{}' a_{pj}a_{qj},$$
$$a_7=\sum_i{}' a_{ip}^2,\qquad a_8=\sum_i{}' a_{iq}^2,\qquad a_9=\sum_i{}' a_{ip}a_{iq},\tag{2.1}$$
$$f_1=a_2+a_1\cos\theta+a_3\sin\theta,$$
$$f_2=-a_2+a_1\cos\theta+a_3\sin\theta,$$
$$f_3=a_3\cos\theta-a_1\sin\theta,$$
$$f_4=\frac{1}{2}(a_4+a_5)+\frac{1}{2}(a_4-a_5)\cos\theta+a_6\sin\theta,$$
$$f_5=\frac{1}{2}(a_4+a_5)-\frac{1}{2}(a_4-a_5)\cos\theta-a_6\sin\theta,$$
$$f_6=a_6\cos\theta-\frac{1}{2}(a_4-a_5)\sin\theta,$$
$$f_7=\frac{1}{2}(a_7+a_8)+\frac{1}{2}(a_7-a_8)\cos\theta+a_9\sin\theta,$$
$$f_8=\frac{1}{2}(a_7+a_8)-\frac{1}{2}(a_7-a_8)\cos\theta-a_9\sin\theta,$$
$$f_9=a_9\cos\theta-\frac{1}{2}(a_7-a_8)\sin\theta,$$
$$b_1=\frac{1}{2}(b_{pq}+b_{qp}),\qquad b_2=\frac{1}{2}(b_{pq}-b_{qp}),\qquad b_3=\frac{1}{2}(b_{qq}-b_{pp}),$$