ON THE CONTRACTIVITY REGION OF RUNGE-KUTTA METHODS*

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Abstract

In this paper we first introduce the definition of contractivity region of Runge-Kutta methods and then examine the general features of the contractivity regions. We find that the intersections of the contractivity region and the axis plane in C^s are always either the whole axis plane or a generalized disk introduced by Dahlquist and Jeltsch. We also define the AN-contractivity and show that it is equivalent to the algebraic stability and can be determined locally in a neighborhood of the origin. However, many implicit methods are only r-circle contractive, but not AN-contractive. A simple bound for the radius r of the r-circle contractive methods is given.

1. Introduction

We shall consider the numerical solution of initial value problems

$$y' = f(x, y), \quad y(0) \quad \text{given} \tag{1.1}$$

where $y, f \in R^s$ or C^s. Assume that f satisfies the following monotonicity condition

$$\mathrm{Re}\,\langle f(x, y) - f(x, z), y - z\rangle \leqslant 0 \quad \text{for } y, z \in R^s \text{ or } C^s, \tag{1.2}$$

where $\langle \cdot, \cdot\rangle$ stands for an arbitrary inner product in C^s, and $\|\cdot\|$ is the corresponding norm. Let y and $\tilde{y}$ be two solutions to (1.1) corresponding to the initial values y_0 and $\tilde{y}_0$ respectively. By condition (1.2) we have

$$\frac{d}{dx}\|y(x) - \tilde{y}(x)\|^2 \leqslant 0 \tag{1.3}$$

which shows that $\|y(x) - \tilde{y}(x)\|$ does not increase when x increases.

The general m-stage Runge-Kutta methods for system (1.1) have the form

$$\begin{cases} Y_i = y_{n-1} + h\sum_{j=1}^{m} a_{ij} f(x_{n-1} + hc_j, Y_j), & i = 1, 2, \cdots, m, \\ y_n = y_{n-1} + h\sum_{j=1}^{m} b_j f(x_{n-1} + hc_j, Y_j), & n = 1, 2, \cdots, \\ c_j = \sum_{k=1}^{m} a_{jk}. \end{cases} \tag{1.4}$$

Given $A = (a_{ij})_{m\times m}$ and $b = (b_1, b_2, \cdots, b_m)^T$, we shall denote the corresponding method (1.4) by $M(A, b)$. In terms of the Kronecker product symbol $\otimes$ it can be written as

$$\begin{cases} Y = 1 \otimes y_{n-1} + hA \otimes I_s F_{n-1}(Y), \\ y_n = y_{n-1} + hb^T \otimes I_s F_{n-1}(Y), \end{cases} \tag{1.5}$$

where I_s is the $s\times s$ identity matrix and

* Received October 13, 1981.

$$Y=\begin{Bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{Bmatrix}, \quad F_{n-1}(Y)=\begin{Bmatrix} f(x_{n-1}+hc_1, Y_1) \\ f(x_{n-1}+hc_2, Y_2) \\ \vdots \\ f(x_{n-1}+hc_m, Y_m) \end{Bmatrix}, \quad \mathbb{1}=\begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix}.$$

In applications it is expected that the numerical methods preserve the contractivity property (1.3) of the differential equation, namely if the computation starts with a slightly perturbed initial value $\tilde{y}_0$, instead of y_0, the obtained solution $\tilde{y}_n$ and the unperturbed solution y_n satisfy

$$\|y_n-\tilde{y}_n\|\leqslant\|y_{n-1}-\tilde{y}_{n-1}\| \quad \text{for } n=1, 2, \cdots. \tag{1.6}$$

Such requirement for the nonlinear problem (1.1) leads to the concept of BN-stability (B-stability for the autonomous problem: $y'=f(y)$, $y(0)=y_0$) introduced by Butcher in [1] and leads to the concept of AN-stability for the linear non-autonomous problem (A-stability for the linear autonomous problem). Another stability criterion named algebraic stability was developed by Butcher[2] and Crouzeix[3], which is significant in the study of BN- and B-stability properties of implicit Runge-Kutta methods. Dahlquist and Jeltsch introduced in [4] a concept of generalized disk contractivity for explicit and implicit Runge-Kutta methods, which is an extension of the AN-and BN-stability that are reasonable only for implicit methods.

In this paper we first introduce the definition of contractivity region of Runge-Kutta methods (implicit or explicit) and then examine the general features of the contractivity region. We find that the intersections of the contractivity region and the axis planes in C^s are always either the whole axis plane or a generalized disk introduced by Dahlquist and Jeltsch[4]. This fact gives some evidence to the concept of generalized disk contractivity. Set $C^-=\{z\in C;\ \mathrm{Re}\, z<0\}$. A method $M(A, b)$ is referred to as "AN-contractive" if its contractivity region contains $(C^-)^m$. We shall show that this property is equivalent to the algebraic stability and can be determined locally in a neighborhood of the origin. However, we shall see that many implicit methods are only r-circle contractive, but not AN-contractive. We shall provide a simple bound for the radius r of the r-circle contractive methods.

2. Contractivity Region

To motivate the definition we consider the following test problem

$$y'=\lambda(x)y, \quad y(0)=y_0, \tag{2.1}$$

where $\lambda: R^+\Rightarrow C$ is a given function and $\mathrm{Re}\,\lambda(x)\leqslant 0$ for $x\in R^+$. Set

$$z_i=h\lambda(x_{n-1}+hc_i), \quad i=1, 2, \cdots, m,$$
$$\zeta=(z_1, z_2, \cdots, z_m),$$
$$Z=\mathrm{diag}(z_1, z_2, \cdots, z_m).$$

For this problem, (1.4) takes the form

$$\begin{cases} Y=y_{n-1}\mathbb{1}+AZY, \\ y_n=y_{n-1}+b^TZY, \end{cases} \tag{2.2}$$

and by substitution $Y=(I_m-AZ)^{-1}(y_{n-1}\mathbb{1})$ we have