ON THE STABILITY OF INTERPOLATION*

HUANG HONG-CI (黄鸿慈)
(*Computing Center, Academia Sinica*)

Abstract

Some definitions on stability of interpolating process are given and then the sufficient and necessary conditions are obtained. On this basis, we conclude that the Lagrange interpolation is unstable, whereas several types of piecewise low order polynomial interpolation are stable. For high order approximation with data on isometric nodes, we recommend the Bernstein approximation owing to its high stability. Some ideas on the relationship between stability and convergence of interpolating process are also presented.

As indicated in practice, the high order polynomial interpolation is unstable, whereas the piecewise low order polynomial interpolation behaves very well. Some of the reasons have been given in [1, p. 12]. This paper will make a theoretical analysis of this problem in detail.

1. The Concept of Stability

By interpolation we mean: On an interval $[a, b]$, given the following infinite node triangle

$$\begin{array}{llll} x_0^0 & & & \\ x_0^1 & x_1^1 & & \\ x_0^2 & x_1^2 & x_2^2 & \\ & \cdots & & \\ x_0^n & x_1^n \cdots\cdots x_n^n & & \\ & \cdots & & \end{array} \tag{1.1}$$

where $a \leqslant x_i^n \leqslant b$ and $x_i^n \neq x_j^n (i \neq j)$, there exists a function set S_n corresponding to each row of nodes $x_i^n (0 \leqslant i \leqslant n)$ such that for the given data of a function $f(x)$ on nodes x_i^n (function values $f(x_i^n)$ and possibly its derivatives $f^{m_i}(x_i^n)$), there exists a unique $\varphi \in S_n$ satisfying

$$f(x_i^n) = \varphi(x_i^n),$$

and possibly,

$$f^{m_i}(x_i^n) = \varphi^{m_i}(x_i^n), \quad m_i \geqslant 1.$$

Then φ is called the interpolating function of f, denoted by

$$\varphi(x) = I_n[f; x].$$

In general, S_n is a linear space and

$$1 \in S_n, \ \forall n. \tag{1.2}$$

* The English version of this paper was received August 9, 1982. The Chinese version of this paper was published in ("计算数学") (*Mathematica Numerica Sinica*), 4: 2 (1982), 193—203.

Obviously, if $\varphi \in S_n$, then

$$\varphi(x) = I_n[\varphi; x]. \tag{1.3}$$

When the given data include $f(x_i^n)$ only and S_n is a polynomial set with degree $\leqslant n$, this is the well-known Lagrange interpolation. When the given data include $f(x_i^n)$ and $f'(x_i^n)$, and S_n is a polynomial set with degree $\leqslant 2n+1$, this is Hermite interpolation. When the given data include function values (or also derivative values) and S_n is a set of piecewise polynomials with some smooth conditions on nodes, this is known as piecewise polynomial interpolation.

We now begin with the case that the given data include function values only. Due to the linearity of the interpolation operator, we give

Definition 1. *The interpolating process is said to be stable respect to the function, if* $\forall \varepsilon > 0$, $\exists \delta$ *such that*

$$\max_{0 \leqslant i \leqslant n} |f(x_i^n)| \leqslant \delta,$$

implies

$$\|I_n[f; x]\|_\infty \leqslant \varepsilon, \quad \forall n.$$

In Definition 1 and henceforward, the symbol $\|f\|_\infty$ stands for $\sup_{a \leqslant x \leqslant b} |f(x)|$.

Let $\{l_i^n(x)\}_{0 \leqslant i \leqslant n}$ be the base of interpolation satisfying

$$\begin{cases} l_i^n(x) \in S_n \\ l_i^n(x_j^n) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases} \end{cases} \tag{1.4}$$

The expression of the basis functions may be simple (for example the Lagrange interpolation) or complex (for example the Hermite interpolation involving data of high order derivatives), or even may not have explicit form (for example the spline interpolation). From (1.4) each interpolating function can be written as

$$I_n[f; x] = \sum_{i=0}^{n} f(x_i^n) l_i^n(x). \tag{1.5}$$

Because of (1.2), (1.3) and (1.5),

$$\sum_{i=0}^{n} l_i^n(x) \equiv 1. \tag{1.6}$$

Denote

$$\lambda_n = \sup_{a \leqslant x \leqslant b} \sum_{i=0}^{n} |l_i^n(x)|. \tag{1.7}$$

Then we have

Lemma 1. *A necessary and sufficient condition for the stability of interpolation is that* λ_n *is bounded for all* n.

Proof. From (1.5) and (1.7) we have

$$\|I_n[f; x]\|_\infty \leqslant \lambda_n \max_{0 \leqslant i \leqslant n} |f(x_i^n)|. \tag{1.8}$$

Thus the sufficient part is obtained.

From (1.7), for any n and arbitrary small $\varepsilon > 0$ there exists ξ_n such that

$$\sum_{i=0}^{n} |l_i^n(\xi_n)| \geqslant \lambda_n - \varepsilon. \tag{1.9}$$

Take

$$f(x_i^n) = \delta \cdot \operatorname{sign} l_i^n(\xi_n), \tag{1.10}$$