FINITE DIFFERENCE METHOD OF THE BOUNDARY PROBLEMS FOR THE SYSTEMS OF GENERALIZED SCHRÖDINGER TYPE*

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§ 1

The nonlinear Schrödinger equation

$$u_t - iu_{xx} + \beta|u|^p u = 0 \tag{1}$$

and the nonlinear Schrödinger system

$$\begin{aligned} &u_t - iu_{xx} + u(\alpha|u|^2 + \beta|v|^2) = 0, \\ &v_t - iv_{xx} + v(\alpha|u|^2 + \beta|v|^2) = 0 \end{aligned} \tag{2}$$

of complex valued functions u and v often appear in the study of problems of physics. These equations and systems may be regarded as the special cases of the system

$$u_t = Au_{xx} + f(u) \tag{3}$$

of real valued functions, where $u(x, t) = (u_1(x, t), \cdots, u_m(x, t))$ is a m-dimensional vector valued unknown function, A is a $m \times m$ non-negatively definite and non-singular constant matrix and $f(u) = (f_1(u), \cdots, f_m(u))$ is a m-dimensional vector valued function of vector variable u. The system (3) may be called the system of generalized Schrödinger type. In [1, 2] the periodic boundary problem and the initial value problem for the system of generalized Schrödinger type of higher order are studied by the method of straight line and the method of Galërkin respectively. In [3] the first boundary value problem for the system (3) is discussed by use of the fixed-point technique and the method of integral estimations. There are many works contribute to the finite difference method for solving the problems of Schrödinger equations.

The purpose of this paper is to solve the boundary problems in rectangular domain $Q_T = \{0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T\}$ for the system (3) of generalized Schrödinger type by means of finite difference method. Assume that the boundary problems (*) take one of the following boundary conditions: the first boundary condition

$$u(0, t) = u(l, t) = 0; \tag{4}$$

the second boundary conditon

$$u_x(0, t) = u_x(l, t) = 0 \tag{5}$$

and the mixed boundary conditions

$$u(0, t) = u_x(l, t) = 0 \tag{6}$$

or

$$u_x(0, t) = u(l, t) = 0. \tag{7}$$

* Received November 1, 1982.

The initial value condition is

$$u(x,\ 0)=\varphi(x), \tag{8}$$

where $\varphi(x)$ is a m-dimensional vector valued initial function, satisfying the appropriate boundary condition $(*)$. We denote any given one of boundary conditions (4), (5), (6) or (7) by the symbol $(*)$.

Let us divide the rectangular domain Q_T into small grids by the parallel lines $x=x_j$ $(j=0,\ 1,\ \cdots,\ J)$ and $t=t_n$ $(n=0,\ 1,\ \cdots,\ N)$, where $x_j=jh,\ t_n=nk,\ Jh=l,\ Nk=T$ $(j=0,\ 1,\ \cdots,\ J;\ n=0,\ 1,\ \cdots,\ N)$. Denote the vector valued discrete function on the grid point $(x_j,\ t_n)$ by v_j^n $(j=0,\ 1,\ \cdots,\ J;\ n=0,\ 1,\ \cdots,\ N)$. Let us construct the finite difference system

$$\frac{v_j^{n+1}-v_j^n}{k}=A\frac{1}{h^2}\Delta_+\Delta_-v_j^{n+1}+f_j^{n+1}, \tag{3$_h$}$$

where $\Delta_+v_j=v_{j+1}-v_j$, $\Delta_-v_j=v_j-v_{j-1}$ and $f_j^{n+1}=f(v_j^{n+1})$. The finite difference boundary conditions are as follows:

$$v_0^n=v_J^n=0; \tag{4$_h$}$$

$$v_1^n-v_0^n=v_J^n-v_{J-1}^n=0; \tag{5$_h$}$$

$$v_0^n=v_J^n-v_{J-1}^n=0; \tag{6$_h$}$$

$$v_1^n-v_0^n=v_J^n=0, \tag{7$_h$}$$

where $n=1,\ 2,\ \cdots,\ N$. The initial condition is as

$$v_j^0=\bar{\varphi}_j,\quad j=0,\ 1,\ \cdots,\ J, \tag{8$_h$}$$

where $\bar{\varphi}_j=\varphi(x_j)$ $(j=0,\ 1,\ \cdots,\ J)$ and $\bar{\varphi}_1=\varphi(0)$ (or $\bar{\varphi}_{J-1}=\varphi(l)$) in the case of boundary condition $v_1^n-v_0^n=0$ (or $v_J^n-v_{J-1}^n=0$). Hence the discrete function $\bar{\varphi}_j$ $(j=0,\ 1,\ \cdots,\ J)$ also satisfies the boundary condition $(*)$.

Now we make the following assumptions for the system (3) of generalized Schrödinger type and the initial vector valued function $\varphi(x)$.

(I) A is a $m\times m$ non-negatively definite and non-singular constant matrix.

(II) The m-dimensional vector valued function $f(u)$ of the vector variable u satisfies the condition of monotonicity

$$(u-v,\ f(u)-f(v))\leqslant b\,|u-v|^2, \tag{9}$$

where b is a constant.

(III) The components of the m-dimensional vector valued initial function $\varphi(x)$ are twice continuously differentiable in $[0,\ l]$. Denote $\varphi(x)\in C^{(2)}([0,\ l])$. And $\varphi(x)$ satisfies the appropriate boundary condition $(*)$.

The scalar product of two vectors u and v is denoted by $(u,\ v)$ and $|u|^2=(u,\ u)$. For the discrete functions $\{u_j\}$ and $\{v_j\}$, we have the symbols $(u,\ v)_h=\sum_{j=0}^{J}u_jv_jh$ and $\|u\|_h^2=(u,\ u)_h$.

§ 2

The finite difference system $(3)_h$ and the finite difference boundary conditions $(*)_h$ can be considered as the nonlinear system of unknown vectors v_j^{n+1} $(j=0,\ 1,\ \cdots,\ J)$, where v_j^n $(j=0,\ 1,\ \cdots,\ J)$ are the known vectors. Now we are going to prove the existence of the solutions v_j^{n+1} $(j=0,\ 1,\ \cdots,\ J)$ for the nonlinear system $(3)_h$ and $(*)_h$.