LOCAL EXPLICIT MANY-KNOT SPLINE HERMITE APPROXIMATION SCHEMES*

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Abstract

If $f^{(i)}(\alpha)$ ($\alpha=a, b, i=0, 1, \cdots, k-2$) are given, then we get a class of the Hermite approximation operator $Qf=F$ satisfying $F^{(i)}(\alpha)=f^{(i)}(\alpha)$, where F is the many-knot spline function whose knots are at points $y_i: a=y_0<y_1<\cdots<y_{k-1}=b$, and $F\in P_k$ on $[y_{i-1}, y_i]$. The operator is of the form $Qf:=\sum_{i=0}^{k-2}[f^{(i)}(a)\phi_i+f^{(i)}(b)\psi_i]$. We give an explicit representation of ϕ_i and ψ_i in terms of B-splines $N_{i,k}$. We show that Q reproduces appropriate classes of polynomials.

1. Introduction

Some authors considered operators of the form $Qf=\sum\lambda_i f N_{i,k}$, where $\{N_{i,k}\}$ is a sequence of B-splines and $\{\lambda_i\}$ is a sequence of linear functionals. The variation diminishing method of Schoenberg ([9], [5], [6]) and the quasi-interpolant of de Boor and Fix are well-known. Such approximation schemes have several important advantages over spline interpolation. They can be constructed directly without matrix inversion, and local error bounds can be obtained naturally. Qi considered the so-called many-knot splines which have many more knots than degrees of freedom and constructed the cardinal spline $Qf=\sum f(x_i)q_{i,k}$, where $q_{i,k}$ is made up of B-splines on a uniform partition, has small support and satisfies $q_{i,k}(x_j)=\delta_{ij}$[7]. Such an approximation operator reproduces appropriate classes of polynomials[8].

The purpose of this paper is to construct a class of many-knot explicit local polynomial spline approximation operators for Hermite interpolation of real-valued functions defined on some interval $[a, b]$.

Let P_k be a set of polynomials of degree less than k, and let

$$a=y_0<y_1<\cdots<y_{k-1}=b. \tag{1.0}$$

We define

$$\hat{S}_k:=\{g: g|_{(y_i, y_{i+1})}\in P_k, \quad i=0, 1, \cdots, k-2\}.$$

$\hat{S}_k$ is the familiar class of polynomial splines of order k with knots at the points $y_i (i=0, 1, \cdots, k-1)$.

Let $\mathscr{F}$ be a linear space of real valued functions on $[a, b]$, and suppose $\mathscr{F}$ contains the class of polynomials P_k. Given $f\in\mathscr{F}$, we construct an approximation $F(\cdot)=Qf(\cdot)$ such that

$$\mathscr{F}^{(l)}(a)=f^{(l)}(a), \ \mathscr{F}^{(l)}(b)=f^{(l)}(b), \quad l=0, 1, \cdots, k-2. \tag{1.1}$$

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In other words, set

$$Qf := \sum_{j=0}^{k-2} f^{(j)}(a)\phi_j(x) + \sum_{j=0}^{k-2} f^{(j)}(b)\psi_j(x); \tag{1.2}$$

suppose ϕ_j, ψ_j satisfying

$$\phi_j^{(i)}(a) = \delta_{ij}, \quad \phi_j^{(i)}(b) = 0, \tag{1.3}$$

$$\psi_j^{(i)}(a) = 0, \quad \psi_j^{(i)}(b) = \delta_{ij}, \quad i, j = 0, 1, \cdots, k-2. \tag{1.4}$$

If ϕ_j and ψ_j are chosen in P_{2k-2}, then the problem above has been considered (see, for instance, [1], [3], [4]), and in this case $F \in P_{2k-2}$ on $[a, b]$.

We will find a many-knot spline $F \in \hat{S}_k$ satisfying (1.1). Such many-knot cardinal splines $\{\phi_j\}$ and $\{\psi_j\}$ are of degree less than k; therefore F is also of degree less than k. We present ϕ_j and ψ_j as explicit representations.

This paper proves that the many-knot spline Hermite approximation operator Q reproduces appropriate classes of polynomials on $[a, b]$.

2. Construction of ϕ_j and ψ_j

Without loss of generality, we assume $a=0$ and $b=1$. First of all we set $k=3$ as an example.

Let ϕ_0, ϕ_1, ψ_0, ψ_1 be piecewise polynomials of degree 2 with knots $0, \frac{1}{2}, 1$ satisfying the following conditions

$$\phi_0(0) = 1, \qquad \phi_1'(0) = 1,$$

$$\phi_0'(0) = \phi_0(1) = \phi_0'(1) = 0, \quad \phi_1(0) = \phi_1(1) = \phi_1'(1) = 0,$$

$$\phi_0\left(\frac{1}{2}+0\right) = \phi_0\left(\frac{1}{2}-0\right), \quad \phi_1\left(\frac{1}{2}+0\right) = \phi_1\left(\frac{1}{2}-0\right),$$

$$\phi_0'\left(\frac{1}{2}+0\right) = \phi_0'\left(\frac{1}{2}-0\right), \quad \phi_1'\left(\frac{1}{2}+0\right) = \phi_1'\left(\frac{1}{2}-0\right),$$

and $\psi_0(x) := \phi_0(1-x)$, $\psi_1(x) := -\phi_1(1-x)$.

Easily one gets

$$\phi_0(x) = \begin{cases} -2x^2+1, & x \in \left[0, \frac{1}{2}\right], \\ 2(x-1)^2, & x \in \left[\frac{1}{2}, 1\right]; \end{cases}$$

$$\phi_1(x) = \begin{cases} -\frac{3}{2}x^2 + x, & x \in \left[0, \frac{1}{2}\right], \\ \frac{1}{2}(x-1)^2, & x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Their graphs are sketched as follows