ON THE EXISTENCE OF FUNCTIONS WITH PRESCRIBED BEST L_1 APPROXIMATIONS*

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Abstract

This paper gives a partial answer to a problem of Rivlin[1] in L_1 approximation.

1. Introduction

In this paper we prove the following ($X\equiv[-1, 1]$)

Theorem. *Let V_1 and V_2 be Chebyshev subspaces of $C(X)$ with dimensions m and $n(m<n)$, respectively. Let $V_1\subset V_2$ and $v_j\in V_j(j=1, 2)$.*

(a) *If the function $v=v_2-v_1$ changes sign at least m times in X, then there exists an $f\in C(X)$ such that v_j is a best L_1 approximation to f from $V_j(j=1, 2)$;*

(b) *If there exists an $f\in C(X)$ such that v_j is a best L_1 approximation to f from V_j $(j=1, 2)$, then v has at least m zeros in $(-1, 1)$.*

This theorem provides a partial answer to a problem of Rivlin[1] in L_1 approximation. However, in the case $m=n-1$ if $v\neq 0$ has at least m zeros in $(-1, 1)$, then none of them can be a double zero and v, in fact, changes sign at least m times. Thus,. we can give the complete answer in this particular case, which is a generalization of the result[2] by the author, and we have

Corollary. Let V_1 and V_2 be Chebyshev subspaces of $C(X)$ with dimensions $n-1$ and $n(n>1)$, respectively. Let $V_1\subset V_2$ and $v_j\in V_j(j=1, 2)$. Then there exists an $f\in C(X)$ such that v_j is a best L_1 approximation to f from $V_j(j=1, 2)$ if and only if the function $v=v_2-v_1$ changes sign at least $n-1$ times in X or is identically zero.

Before proving the theorem we introduce some notation:

$$Z_+(g)=\{x\in X: g(x)>0\},$$

$$Z_-(g)=\{x\in X: g(x)<0\},$$

$$Z(g)=\{x\in X: g(x)=0\},$$

$$M(E)=\text{the Lebesgue measure of the set } E.$$

2. Proof of Part (a) of the Theorem

Let v change sign at points x^k, $k=1, 2, \cdots, l$ $(l\geqslant m)$,

* Received December 17, 1982.

1) This work was supported by a grant to Professor C. B. Dunham from the Natural Sciences and Engineering Research Council of Canada.

$$-1=x^0<x^1<\cdots<x^l<x^{l+1}=1.$$

By Lemma 2 in [3] there exist points

$$x^k=x_0^k<x_1^k<\cdots<x_n^k<x_{n+1}^k=x^{k+1},\quad k=0,\ 1,\ \cdots,\ l,$$

such that

$$\sum_{i=0}^{n}(-1)^i\int_{x_i^k}^{x_{i+1}^k}udx=0,\ \forall u\in V_2,\quad k=0,\ 1,\ \cdots,\ l. \tag{1}$$

Write $n_j=\left[\frac{1}{2}(n+1-j)\right]$(the integral part of $\frac{1}{2}(n+1-j)$), $j=1,\ 2$ and denote

$$G_j^k=\bigcup_{i=0}^{n_j}[x_{2i+j-1}^k,\ x_{2i+j}^k],\quad k=0,\ 1,\ \cdots,\ l,\ j=1,\ 2,$$

$$G_j=\bigcup_{k=0}^{[l/2]}G_j^{2k},\quad j=1,\ 2,$$

$$G_j^*=\bigcup_{k=0}^{[\frac{1}{2}(l-1)]}G_j^{2k+1},\quad j=1,\ 2,$$

$$H_i^k=(x_i^k-h,\ x_i^k+h)\cap(G_2\cup G_2^*),\quad i=1,\ 2,\ \cdots,\ n,\ k=0,\ 1,\ \cdots,\ l,$$

$$H=\bigcup_{k=0}^{l}\bigcup_{i=1}^{n}H_i^k\cap G_2,$$

$$H^*=\bigcup_{k=0}^{l}\bigcup_{i=1}^{n}H_i^k\cap G_2^*,$$

where $0<h<\frac{1}{2}\min\limits_{\substack{1<i<n\\0<k<l}}(x_{i+1}^k-x_i^k)$ will be defined later. With this notation (1) becomes

$$\int_{G_1^k}udx=\int_{G_2^k}udx,\ \forall u\in V_2,\quad k=0,\ 1,\ \cdots,\ l.$$

Whence

$$\int_{G_1}udx=\int_{G_2}udx,\ \int_{G_1^*}udx=\int_{G_2^*}udx,\ \forall u\in V_2. \tag{2}$$

Now put

$$f(x)=\begin{cases}v_2(x),\ x\in G_1\cup G_1^*,\\ v_1(x),\ x\in(G_2\cup G_2^*)\setminus(H\cup H^*),\\ \text{a continuous function on } H_i^k \text{ lying strictly between } v_1 \text{ and } v_2\\ \quad\text{almost everywhere on } \overline{H}_i^k,\quad i=1,\ 2,\ \cdots,\ n,\ k=0,\ 1,\ \cdots,\ l.\end{cases}$$

It is easy to see that $f\in C(X)$. Now take $x^*<x^1$ such that $v(x^*)\neq 0$ and let $s=\operatorname{sgn}\ v(x^*)$. Thus

$$\operatorname{sgn}(f(x)-v_1(x))=\begin{cases}s,\quad x\in G_1\cup H,\\ -s,\quad x\in G_1^*\cup H^*,\\ 0,\quad x\in(G_2\cup G_2^*)\setminus(H\cup H^*)\end{cases}$$

and

$$\operatorname{sgn}(f(x)-v_2(x))=\begin{cases}-s,\quad x\in G_2,\\ s,\quad x\in G_2^*,\\ 0,\quad x\in G_1\cup G_1^*\end{cases}$$

are valid almost everywhere.