THE CONVERGENCE OF THE SPECTRAL SCHEME FOR SOLVING TWO-DIMENSIONAL VORTICITY EQUATIONS*

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Much work has been done for the spectral scheme of the P. D. E. (see[1]). The author proposed a technique to prove the strict error estimation of the spectral scheme for the K. D. V.-Burgers equation[2]. In this paper, the technique is generalized to two-dimensional vorticity equations. Under some conditions, the error estimation implies the convergence. The more smooth the solution of the vorticity equations, the more accurate the approximate solution.

I. The Scheme

Let $H(x_1, x_2, t)$ and $\Psi(x_1, x_2, t)$ be the vorticity and stream function respectively. $f_i(x_1, x_2, t)$ are given. All of them have the period 2π for variables x_1 and x_2.

Let
$$Q=\text{set}[(x_1, x_2)/-\pi\leqslant x_1,\ x_2\leqslant\pi],$$
$$F_p(Q)=\{\varphi/\varphi\in H^p,\ \varphi(x_1, x_2)=\varphi(x_1+2\pi, x_2)=\varphi(x_1, x_2+2\pi)\},$$
$$J(H, \Psi)=\frac{\partial\Psi}{\partial x_2}\frac{\partial H}{\partial x_1}-\frac{\partial\Psi}{\partial x_1}\frac{\partial H}{\partial x_2}.$$

We consider the following problem
$$\begin{cases}\dfrac{\partial H}{\partial t}+J(H, \Psi)-\nu\nabla^2H=f_1, & \text{in } Q\times(0, T],\\ -\nabla^2\Psi=H+f_2, & \text{in } Q\times[0, T],\\ H(x_1, x_2, 0)=H_0(x_1, x_2), & \text{in } Q,\end{cases}\tag{1}$$
where ν is a nonnegative constant. We suppose
$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(H_0(x_1, x_2)+f_2(x_1, x_2, t))dx_1dx_2+\int_0^t\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f_1(x_1, x_2, t')dx_1dx_2dt'=0.$$

Let
$$(\eta(t), \xi(t))=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\eta(x_1, x_2, t)\xi(x_1, x_2, t)dx_1dx_2.$$

To fix $\Psi(x_1, x_2, t)$, we require
$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Psi(x_1, x_2, t)dx_1dx_2=0, \quad t\in[0, T].\tag{2}$$

We take the solution of (1) as follows:

* Received December 30, 1982. The Chinese version was received January 23, 1981.

$$\begin{cases} \left(\frac{\partial H}{\partial t}, \varphi\right)+(J(H, \Psi), \varphi)+\nu \sum_{j=1}^{2}\left(\frac{\partial H}{\partial x_j}, \frac{\partial \varphi}{\partial x_j}\right)=(f_1, \varphi), \\ \sum_{j=1}^{2}\left(\frac{\partial \Psi}{\partial x_j}, \frac{\partial \varphi}{\partial x_j}\right)=(H+f_2, \varphi), \end{cases} \tag{3}$$

where $\varphi \in F_1(Q)$. (3) is supposed to have a unique solution. Let

$$x=(x_1, x_2),\ l=(l_1, l_2),\ |l|=\sqrt{l_1^2+l_2^2},\ lx=l_1x_1+l_2x_2.$$

Put

$$H(x, t)=\sum_{|l|=0}^{\infty} H_l(t)e^{ilx},$$

$$\Psi(x, t)=\sum_{|l|=0}^{\infty} \Psi_l(t)e^{ilx},$$

$$f_j(x, t)=\sum_{|l|=0}^{\infty} f_{j,l}(t)e^{ilx}, \qquad j=1, 2,$$

$$H^{(n)}(x, t)=\sum_{|l|<n} H_l(t)e^{ilx},$$

$$\Psi^{(n)}(x, t)=\sum_{|l|<n} \Psi_l(t)e^{ilx},$$

$$f_j^{(n)}(x, t)=\sum_{|l|<n} f_{j,l}(t)e^{ilx}; \qquad j=1, 2,$$

and

$$R^{(n)}(H)=H(x, t)-H^{(n)}(x, t),$$

$$R^{(n)}(\Psi)=\Psi(x, t)-\Psi^{(n)}(x, t),$$

$$R^{(n)}(f_j)=f_j(x, t)-f_j^{(n)}(x, t), \qquad j=1, 2.$$

We assume that H, Ψ and f_j are so smooth that when $n\to\infty$, $R^{(n)}\left(\frac{\partial H}{\partial x_j}\right)$, $R^{(n)}\left(\frac{\partial \Psi}{\partial x_j}\right)$, $R^{(n)}(f_1)$ and $R^{(n)}(f_2)$ tend to zero in $Q\times[0, T]$.

Let τ be the mesh spacing of variable t and be sufficiently small,

$$\eta_t(x, K\tau)=\frac{1}{\tau}[\eta(x, K\tau+\tau)-\eta(x, K\tau)], \quad K\geqslant 0.$$

Let $\eta^{(n)}(x, t)$ and $\psi^{(n)}(x, t)$ denote the approximation of $H^{(n)}(x, t)$ and $\Psi^{(n)}(x, t)$ respectively

$$\eta^{(n)}(x, K\tau)=\sum_{|l|<n} \eta_l^{(n)}(K\tau)e^{ilx},$$

$$\psi^{(n)}(x, K\tau)=\sum_{|l|<n} \psi_l^{(n)}(K\tau)e^{ilx}.$$

$\delta\geqslant 0$ and $\sigma\geqslant 0$ are parameters, $\varphi_l=e^{ilx}$.

The spectral scheme for solving (1) is the following

$$\begin{cases} (\eta_t^{(n)}(K\tau), \varphi_l)+(J(\eta^{(n)}(K\tau)+\delta\tau\eta_t^{(n)}(K\tau), \psi^{(n)}(K\tau)), \varphi_l) \\ \quad +\nu\sum_{j=1}^{2}\left(\frac{\partial}{\partial x_j}(\eta^{(n)}(K\tau)+\sigma\tau\eta_t^{(n)}(K\tau)), \frac{\partial \varphi_l}{\partial x_j}\right) \\ \quad =(f_1^{(n)}(K\tau), \varphi_l), \quad |l|\leqslant n, \quad 0\leqslant K\tau\leqslant T, \\ \sum_{j=1}^{2}\left(\frac{\partial}{\partial x_j}(\psi^{(n)}(K\tau)), \frac{\partial\varphi_l}{\partial x_j}\right)=(\eta^{(n)}(K\tau)+f_2^{(n)}(K\tau), \varphi_l),\ |l|\leqslant n, \quad 0\leqslant K\tau\leqslant T, \\ \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\psi^{(n)}(x, K\tau)dx_1dx_2=0, \quad 0\leqslant K\tau\leqslant T, \\ \eta^{(n)}(x, 0)=H_0^{(n)}(x), \quad x\in Q. \end{cases} \tag{4}$$

Clearly if $\delta=\sigma=0$, and $\eta_l^{(n)}(K\tau)$ and $\psi_l^{(n)}(K\tau)$ are known, we can calculate