

LINEAR FINITE ELEMENT APPROXIMATIONS FOR THE TIMOSHENKO BEAM AND THE SHALLOW ARCH PROBLEMS^{*1)}

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Abstract

In this paper we discuss the linear finite element approximations for the Timoshenko beam and the shallow arch problems with shear dampening and reduced integration. We derive directly the optimal order error estimates uniformly with the small thickness parameter, without relying on the theory of saddle point problems.

Key words: Timoshenko beam, Shallow arch, Shear dampening, Reduced integration.

1. Introduction

In this paper we examine the linear finite element discretization of the Timoshenko beam and the shallow arch problems. The thickness of beam and arch appears parametrically in the model and locking phenomenon may occur due to the small parameter. In [1] Arnold proposed reduced integration formulation for the Timoshenko beam and in [6] Reddy applied the method for the shallow arch problem. In these papers, the problems are analyzed based on the equivalence of reduced integration formulation and a mixed formulation, and thus a major role is played by the theory of saddle point problems.

In this paper we modify the formulation in [1] with shear dampening for the Timoshenko beam and in [6] with shear and axial dampening for the shallow arch problem. We prove directly the uniform convergence with respect to small parameter without relying on the saddle point theory. The proofs of error estimates are elementary, without relying on the theory of the saddle point problem. In the next section, we propose and analyze a family of linear element schemes to solve a Timoshenko beam problem. Additionally, we also derive the explicit formulation of the exact solution, which can be used to evaluate the performance of the numerical methods. In Section 3, we propose and analyze a family of linear element schemes to solve a shallow arch problem.

2. The Timoshenko Beam

2.1. The Model

Following [1], we consider the following variational formulation for a Timoshenko beam model problem: find $(\phi, w) \in H_0^1(I) \times H_0^1(I)$ such that

$$(\phi', \psi') + \frac{1}{\varepsilon}(\phi - w', \psi - v') = (g, v), \quad \forall (\psi, v) \in H_0^1(I) \times H_0^1(I), \quad (2.1)$$

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where $H_0^1(I)$ is the usual Sobolev space, $I = (0, 1)$, (\cdot, \cdot, \cdot) is the $L^2(I)$ inner product, and the parameter $0 < \varepsilon \ll 1$. In the model (2.1), ϕ, w and ε are proportional to the rotation, the vertical displacement and the thickness of the beam, respectively. Denote the shear strain variable

$$\sigma = \frac{1}{\varepsilon}(\phi - w'). \quad (2.2)$$

It is proved in [1] that there exists a unique solution of (2.1) and there is a constant C independent of g and ε such that

$$\|\phi\|_2 + \|w\|_2 + \|\sigma\|_1 \leq C\|g\|_0. \quad (2.3)$$

It is well known that the locking phenomenon occurs when the standard finite element method is applied to the equation (2.1) directly. An effective approach to eliminate the locking phenomenon is the use of the reduced integration technique, see [1]. This method is based on the equivalent formulation of the mixed finite element method. Thus the error estimates were obtained by the saddle point theory. Here we will modify the formulation with shear dampening and obtain the uniform convergence with respect to the small parameter directly.

2.2. Linear Element Scheme with Shear Dampening

First we assume that the interval $[0, 1]$ is partitioned into subintervals $I_e = [x_e, x_{e+1}]$, $0 \leq e \leq N-1$,

$$0 = x_0 < x_1 < \cdots < x_N = 1.$$

Denote $h_e = x_{e+1} - x_e$, $h = \max_e h_e$, and suppose that the mesh refinements are quasi-uniform in the sense that there exists a constant $\theta > 0$ such that $\min_e h_e / \max_e h_e \geq \theta$. Define the linear finite element space

$$W_h = \{v \in H_0^1(I) : v|_{I_e} \in P_1(I_e), e = 0, 1, \dots, N-1\} \quad (2.4)$$

and an auxiliary space

$$Q_h = \{q \in L^2(I) : q|_{I_e} \in P_0(I_e), e = 0, 1, \dots, N-1\}, \quad (2.5)$$

where $P_k(I_e)$ denotes the space of all the polynomials of degree less than or equal to k on the interval I_e .

The approximation problem is to find $(\phi_h, w_h) \in V_h \times V_h$ such that

$$(\phi'_h, \psi'_h) + \frac{1}{\varepsilon + \alpha_0 h^2} (\pi_h(\phi_h - w'_h), \pi_h(\psi_h - v'_h)) = (g, v_h), \quad \forall (\psi_h, v_h) \in V_h \times V_h, \quad (2.6)$$

where $\pi_h : L^2(I) \rightarrow Q_h$ is an orthogonal projection operator, i.e., for $q \in L^2(I)$,

$$\pi_h q|_{I_e} = \frac{1}{h_e} \int_{I_e} q dx, \quad (2.7)$$

and $\alpha_0 > 0$ is a constant independent of ε and h . Obviously, (2.6) has an unique solution.

Remark 2.1. The scheme is actually used the one-point Gaussian quadrature to compute the term $(\phi - w', \psi - v')$. We write the operator π_h for convenience of notation.

Remark 2.2. $\alpha_0 = 0$ is the case of the formulation of [1] and $\alpha_0 = \frac{1}{12}$ is the same as Petrov-Galerkin formulation in [5] with a slightly different right-hand side term. Compared with [4], we remove the bubble function.

2.3. Error Estimates

We introduce an approximation of the shear strain by the formula

$$\sigma_h = (\varepsilon + \alpha_0 h^2)^{-1} \pi_h(\phi_h - w'_h). \quad (2.8)$$