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ON DISCRETE PROJECTION AND NUMERICAL BOUNDARY CONDITIONS FOR THE NUMERICAL SOLUTION OF THE UNSTEADY INCOMPRESSIBLE NAVIER-STOKES EQUATIONS*1)

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Abstract

The unsteady incompressible Navier-Stokes equations are discretized in space and studied on the fixed mesh as a system of differential algebraic equations. With discrete projection defined, the local errors of Crank Nicholson schemes with three projection methods are derived in a straightforward manner. Then the approximate factorization of relevant matrices are used to study the time accuracy with more detail, especially at points adjacent to the boundary. The effects of numerical boundary conditions for the auxiliary velocity and the discrete pressure Poisson equation on the time accuracy are also investigated. Results of numerical experiments with an analytic example confirm the conclusions of our analysis.

Key words: Differential algebraic equations, Discrete projection, Numerical boundary conditions.

1. Introduction

Let us consider the unsteady incompressible Navier–Stokes equations (INSE)

$$\frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{w} \cdot grad)\boldsymbol{w} + grad\ p = \frac{1}{\mathsf{Re}} div\ grad\ \boldsymbol{w} + \mathbf{f} \tag{1.1}$$

$$div\ \boldsymbol{w} = 0 \tag{1.2}$$

on a two–dimensional rectangular region Ω with boundary $\partial\Omega$. Here $\boldsymbol{w} = (u, v)^T$ is the velocity vector; p is the pressure; and $\mathbf{f}$ a known vector function of $x, y,$ and t. The initial condition is given as

$$\boldsymbol{w}|_{t=0} = \mathbf{w}^0 \quad \text{on } \Omega \tag{1.3}$$

satisfying (1.2). We are concerned mainly with the solid wall boundary condition

$$\boldsymbol{w} = \mathbf{w}_B \quad \text{on } \partial\Omega \quad \text{satisfying } \oint_{\partial\Omega} \mathsf{w}_{Bn} ds = 0 \tag{1.4}$$

The difficulty in the numerical solution of the above problem lies in that (1.1) and (1.2) are partial differential equations with constraint; i.e., the system of equations is not entirely evolutionary. The projection methods of Temam [23], Chorin [2], and van Kan [25] have been

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widely used and have proven to be most efficient for this type of problems. However, it is not always well understood and many of its problems need further investigation and rigorous analysis. Historically, Temam [23] gave the convergence proofs for the proposed methods with full discretization, but only Chorin [2] gave the error estimate for his proposed method with full discretization. Then with recent interest, great progress has been made in the study of errors of projection methods; for example: the works of E and Liu [4, 5], Orszag, Israeli, and Deville [16], Shen [20, 21], Rannacher [19], and Hou and Wetton [9, 26]. Significant as these mathematical papers are, many analyses are done for the Stokes equations or for the semi-discrete INSE (space continuous), and some with explicit approximation for convection. It is the author's opinion that much work is required before the error estimation of a projection method for a fully discretized INSE become easily accessible to the computational community.

It has been the author's attempt to contribute in this direction using simple mathematical tools familiar to the computational fluid dynamics community. With spatial discretization on a *fixed* mesh, the INSE become a system of differential algebraic equations (DAE), for which the local errors of a numerical method can be quite different from its counterpart for the ordinary differential equation, see [7] for example. Further errors are introduced with the projection method for the system derived from INSE. The authors [11, 12] defined discrete projection with the minimum requirement as that needed for the projection step in numerical solution of the system from INSE. With the projection operators, the derivation of local errors of the velocity and the pressure gradient for projection methods becomes straightforward with Taylor series. This paper studies in particular the fully implicit (for convection and viscosity) Crank Nicholson (CN) schemes, mainly CN2 to be described below, and three projection methods: pressure correction (PC) studied thoroughly by van Kan in [25], pressure (PR) of the earlier projection papers and the present version here of Kim and Moin [15], and component-consistent pressure correction (CCPC) proposed by the authors in [11] for its approximate preservation of component-consistency under projection. This version has been used by Bell, Colella, and Glaz with a different projection procedure in [1], and interpreted as the present version by [4].

The global errors on the fixed mesh follow from the local errors, as for the general DAE [11], with correct interpretation of the assumption that the right hand side functions have bounded derivatives in some closed region of our interest; but convergence for a finite time interval is almost trivial, as it is for the ordinary differential equations, and gives little information to problems of INSE as partial differential equations. To gain some insight into this type of problems, the local errors are analyzed with approximate factorization (AF) of relevant matrices as Yanenko [27], Perot [17, 18], and the author [10], with special attention to the discrete approximation on points adjacent to the boundary. From our analysis we can see, for example, the reason why an improvement in the numerical boundary condition (NBC) for the auxiliary velocity over just (1.4) can lead to an increase of an order in the accuracy of the velocity, e.g. the Kim and Moin method in [15]. Also several NBCs for the auxiliary velocity and the discrete pressure Poisson equation frequently stated in literature will be investigated and clarified in terms of discrete projection.

In Section 2, the discrete projection will be stated and two Crank Nicholson (CN) schemes for the DAE formed from INSE will be given. Three projection methods based on these schemes: PC, PR, and CCPC will be described and their local errors of the velocity and the pressure gradient will be briefly derived in Section 3. Then these errors will be studied more carefully with the AF method in Section 4. Several NBCs frequently stated in literature will be summarized in terms of discrete projection in Section 5. Finally, in Section 6, the results of numerical experiment with an analytic example, on the staggered mesh for simplicity, will be given. These results confirm the conclusions of our analysis.

Here a word on the notation of this paper is in order: boldface ($\boldsymbol{Z}$) denotes a "double"