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A NEW STABILIZED FINITE ELEMENT METHOD FOR SOLVING THE ADVECTION–DIFFUSION EQUATIONS*[1)]

Huo-yuan Duan
(*Institute of Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

This paper is devoted to the development of a new stabilized finite element method for solving the advection–diffusion equations having the form $-\kappa\,\Delta\,u+\underline{a}\bullet\underline{\nabla}\,u+\sigma\,u=f$ with a zero Dirichlet boundary condition. We show that this methodology is coercive and has a uniformly optimal convergence result for all mesh–Peclet number.

Key words: Advection–diffusion equation, Stabilized finite element method.

1. Introduction

Consider the advection–diffusion equation

$$\begin{cases} -\kappa\,\Delta\,u+\underline{a}\bullet\underline{\nabla}\,u+\sigma\,u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.1}$$

in a bounded polygonal domain $\Omega\subset I\!R^2$ with the boundary $\partial\Omega$, where $0<\kappa\ll 1$ is the diffusion parameter, $\sigma>0$ is a given positive constant, $\underline{a}(x)$ is a given vector field representing the flow with $\underline{\nabla}\bullet\underline{a}=0$ in Ω, and $f\in L^2(\Omega)$ is a given source function. The term $\sigma\,u$ is usually obtained by the time discretization of the nonstationary advection–diffusion equation arising from mathematical and engineering problems, so the item σ takes it form as $1/\triangle\,t$ with $\triangle\,t<1$ being the time step. Generally speaking, σ is comparatively large, and when $\triangle\,t$ or κ tends to zero, a boundary layer region may be present near the boundary.

It is now well known that the standard Galerkin method solving (1.1) often causes a bad numerical solution when the balance among the three parameters σ,κ and $\underline{a}$ is losing. The goal of stabilized finite element methods established in recent decade, e.g. see [5] [8] [10], etc., is to seek for some good approximating solutions of (1.1) on which the effects emanating from the disturbance among σ,κ and $\underline{a}$ can be cut down as much as possible. In [4] [5] [7] [8] [11], some stablized finite element methods with an additional mesh–dependent perturbation bilinear term were proposed, therein a good approximating result was obtained. [5] studied a stabilized method based on local bubble functions for (1.1) with $\sigma=0$ or $\underline{a}=0$, which deduced an optimal error estimation including higher order elements, independent of σ and κ for the case with $\underline{a}=0$, and independent of mesh–Peclet number for the case with $\sigma=0$.

In [1], a bubble–enriching method for advection–diffusion problem without the zeroth order term $\sigma\,u$ is analyzed in details. However, the bubble–enriching method is not fit for the advection dominated case. For this reason, some special local bubble functions are needed, e.g. see [2] [10], e.t., which are usually very difficult to construct. As for the fact that the bubble–enriching method often deduces a stabilized method associated with problem (1.1), it may be also referred to [11] for a heuristic observation.

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Now we consider a general model (1.1). Firstly by defining a stabilizing parameter and by adding a suitable mesh–dependent bilinear form, we design a finite element approximation for (1.1). Next, the coerciveness of the new formulation is shown, and finally the optimal error estimates for all mesh–Peclet number are obtained, including the L^2–norm, and the higher order elements for triangle and quadrilateral partitions of the domain. If introducing a mesh–Peclet number, it can be seen that our results may result in those of [5] [8].

The rest of this paper is outlined as follows. In section 2, the stabilized finite element formulation for (1.1) is described and the coerciveness of this method is investigated. The section 3 is devoted to a general error analysis. In the last section, every case of (1.1) is discussed, according to σ, κ and $\underline{a}$. Sharp error estimates are obtained.

In what follows, for simplicity we shall use C (or $C_i, i = 1, 2, \cdots$) to stand for different constant at different occurence, and they are all independent of $\sigma, \kappa, \underline{a}$ and the mesh size h.

2. Problem Fzormulation

For convenience, we rewrite (1.1)

$$\begin{cases} -\kappa\,\Delta\, u + \underline{a} \bullet \underline{\nabla}\, u + \sigma\, u & = \quad f \qquad \text{in } \Omega \\ \qquad\qquad\qquad\qquad u & = \quad 0 \qquad \text{on } \partial\Omega \end{cases} \tag{2.1}$$

The standard Galerkin variational problem is to find $u \in H_0^1(\Omega)$ such that

$$B(u, v) = (f, v)_0 \quad \forall\, v \in H_0^1(\Omega) \tag{2.2}$$

where

$$B(u, v) = (\sigma\, u, v)_0 + (\underline{a} \bullet \underline{\nabla}\, u, v)_0 + (\kappa \underline{\nabla}\, u, \underline{\nabla}\, v)_0 \tag{2.3}$$

The discrete version of problem (2.2) consists of finding $u_h \in U_h \subset H_0^1(\Omega)$ such that

$$B(u_h, v) = (f, v)_0 \quad \forall\, v \in U_h \tag{2.4}$$

where

$$U_h = \{v \in H_0^1(\Omega) \cap C(\Omega) \,|\, v_K \in R_m(K), K \in \mathcal{E}_h\} \tag{2.5}$$

with $R_m(K) = P_m(K)$ or $Q_m(K)$ corresponding to the partition being triangle or quadrilateral, and $m \geq 1$, P_m, Q_m are the usual finite element subspaces depicted in [3]. $\mathcal{E}_h$ is the regular partition of the domain Ω, which is supposed to be a polygonal bounded region as usual. Also, $C(\Omega)$ is the space of continuous functions in Ω, and $H_0^1(\Omega)$ is the Hilbert space of functions, taking their values as zero along the boundary $\partial\Omega$, which, together with their first–order derivatives, are square–integrable.

For each $K \in \mathcal{E}_h$, define

$$\tau_{K,\alpha} = \frac{h_K^2}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K} \tag{2.6}$$

with $\alpha > 0$ to be determined later. Here h_K is the element parameter for $K \in \mathcal{E}_h$, and

$$[\underline{a}]_K = \sup_{x \in K} |\underline{a}(x)|_p \tag{2.7}$$

with $|\underline{a}(x)|_p = (\sum_{i=1,2} |a_i(x)|^p)^{\frac{1}{p}}$ for $1 \leq p \leq \infty$ and $|\underline{a}(x)|_\infty = \max_{i=1,2} |a_i(x)|$ for $p = \infty$ being a norm in the Euclidean two dimensional space IR^2. Now, a stabilizing bilinear form is introduced as follows.

$$T(u, v) = -\alpha \sum_{K \in \mathcal{E}_h} \tau_{K,\alpha}\, (\sigma\, u + \underline{a} \bullet \underline{\nabla}\, u - \kappa\, \Delta\, u, \sigma\, v - \underline{a} \bullet \underline{\nabla}\, v)_{0,K} \tag{2.8}$$