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# RELAXED ASYNCHRONOUS ITERATIONS FOR THE LINEAR COMPLEMENTARITY PROBLEM*1)

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**Abstract**

We present a class of relaxed asynchronous parallel multisplitting iterative methods for solving the linear complementarity problem on multiprocessor systems, and set up their convergence theories when the system matrix of the linear complementarity problem is an H-matrix with positive diagonal elements.

*Key words*: Linear complementarity problem, Matrix multisplitting, Relaxation method, Asynchronous iteration, Convergence theory.

## 1. Introduction

Consider the large sparse Linear Complementarity Problem (LCP):

$$Mz + q \geq 0, \quad z \geq 0, \quad z^T(Mz + q) = 0,$$

where $M = (m_{kj}) \in L(R^n)$ is a given real matrix and $q = (q_k) \in R^n$ a given real vector. This problem arises in many areas of scientific computing. For example, it arises from problems in (linear and) convex quadratic programming, the problem of finding a Nash equilibrium point of a bimatrix game (e.g., Cottle and Dantzig[5] and Lemke[13]), and also a number of free boundary problems of fluid mechanics (e.g., Cryer[8]). There have been a lot of researches on the approximate solution of the LCP in the literature, e.g., Cottle and Sacher[7], Cottle, Golub and Sacher[6], Mangasarian[14], Mangasarian and De Leone[15] and De Leone and Mangasarian[9]. These works, besides presenting efficient iterative methods, afforded feasible ways and essential techniques for studying the numerical solution of the LCP.

To solve the LCP on high-speed multiprocessor systems, recently, Bai and Evans[2] and Bai, Evans and Wang[3] presented a class of relaxed parallel iterative methods. These methods are based upon several splittings of the system matrix $M \in L(R^n)$, as well as the equivalence of the LCP to a fixed-point system, and they have many advantages such as high parallelism, strong generality and extensive applicability.

In accordance with the principle of sufficiently using the delayed information, and by making use of both the matrix multisplitting and the successive overrelaxation techniques, in this paper we propose a class of new relaxed asynchronous iterations for solving the LCP on the MIMD

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systems. These new methods decrease the exchange frequencies of the information among the processors and increase the efficient numerical computations of each processor. Moreover, mutual waits among the processors of the multiprocessor system are thoroughly avoided. In addition, since two relaxation sweeps are induced within each iteration, and each sweep possibly includes its own pair of relaxation parameters, these new methods can cover all the known and a lot of novel practical and efficient relaxed asynchronous parallel methods in the sense of multisplitting. Following suitable adjustments of the relaxation parameters, the convergence properties of this class of new asynchronous parallel multisplitting relaxation methods can be greatly improved. These are the advantages of our new methods over the known ones discussed in [2] and [3]. When the system matrix $M \in L(R^n)$ is an H-matrix with positive diagonal elements, we establish the convergence theories of these new methods under proper constraints on both the multiple splittings and the relaxation parameters.

This paper affords efficient method models and necessary convergence theories for the asynchronous parallel multisplitting relaxation iterations for solving the LCP on MIMD multiprocessor systems. Essentially, it is an extension of the work of Bai and Evans in [2]; and is also a development of those of Bai, Evans and Wang in [3].

## 2. Preliminaries

We first briefly describe the notations. Let $C = (c_{kj})$ be an $n \times n$ matrix. By $diag(C)$ we denote the $n \times n$ diagonal matrix coinciding in its diagonal with $C$. For $A = (a_{kj})$, $B = (b_{kj}) \in L(R^n)$, we write $A \leq B$ if $a_{kj} \leq b_{kj}$ holds for all $k, j = 1, 2, \cdots, n$. We call $A$ nonnegative if $A \geq 0$. This definition carries immediately over to vectors by identifying them with $n \times 1$ matrices. In particular, we call the vector $x \in R^n$ positive (writing $x > 0$) if all its entries are positive. By $|A| = (|a_{kj}|)$ we define the absolute value of $A \in L(R^n)$; it is a nonnegative $n \times n$ matrix satisfying $|AB| \leq |A||B|$ for any $B \in L(R^n)$. We denote by $\langle A \rangle = (\alpha_{kj})$ the $n \times n$ comparison matrix of $A \in L(R^n)$ where $\alpha_{kj} = |a_{kk}|$ for $k = j$ and $\alpha_{kj} = -|a_{kj}|$ for $k \neq j$, $k, j = 1, 2, \cdots, n$. We call $A = (a_{kj}) \in L(R^n)$ an M-matrix if it is nonsingular with $a_{kj} \leq 0$ for $k \neq j$ and $A^{-1} \geq 0$. We call it an H-matrix if $\langle A \rangle$ is an M-matrix. Note that an H-matrix is nonsingular, and has the properties that $|A^{-1}| \leq \langle A \rangle^{-1}$ and $\rho(|D_A|^{-1}|B_A|) < 1$, where $D_A = diag(A)$, $B_A = D_A - A$ and $\rho(\bullet)$ the spectral radius of a matrix. If $x \in R^n$, $x_+$ is used to denote the vector with elements $(x_+)_j = \max\{0, x_j\}$, $j = 1, 2, \cdots, n$. For any $x, y \in R^n$, there hold: (a) $(x + y)_+ \leq x_+ + y_+$; (b) $x_+ - y_+ \leq (x - y)_+$; (c) $|x| = x_+ + (-x)_+$; and (d) $x \leq y$ implies $x_+ \leq y_+$.

It is well-known that the LCP can be equivalently transformed to a fixed-point problem. More concretely, we have the following conclusion:

**Lemma 2.1.** *(see [2, 14]) A vector $z \in R^n$ solves the LCP if and only if it satisfies*

$$z = (z - \Phi(Mz + q))_+ ,$$

*where $\Phi = diag(\varphi_1, \varphi_2, \cdots, \varphi_n) \in L(R^n)$ is any positive diagonal matrix.*

Since a fixed point equation readily leads to an iterative method, Lemma 2.1 then affords one basis for the establishments of some efficient and practical iterative methods for solving the LCP; see, e.g., [1]-[3], [6]-[9], [13]-[15] and references therein. Moreover, the existence and uniqueness of the solution of the LCP can be directly concluded from Lemma 2.1.

**Lemma 2.2.** *(see [2]) Let $M \in L(R^n)$ be an H-matrix with positive diagonal elements. Then the LCP has a unique solution $z^* \in R^n$.*