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MULTIGRID METHODS FOR THE GENERALIZED STOKES EQUATIONS BASED ON MIXED FINITE ELEMENT METHODS*

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Abstract

Multigrid methods are developed and analyzed for the generalized stationary Stokes equations which are discretized by various mixed finite element methods. In this paper, the multigrid algorithm, the criterion for prolongation operators and the convergence analysis are all established in an abstract and element–independent fashion. It is proven that the multigrid algorithm converges optimally if the prolongation operator satisfies the criterion. To utilize the abstract result, more than ten well–known mixed finite elements for the Stokes problems are discussed in detail and examples of prolongation operators are constructed explicitly. For nonconforming elements, it is shown that the usual local averaging technique for constructing prolongation operators can be replaced by a computationally cheaper alternative, random choice technique. Moreover, since the algorithm and analysis allows using of nonnested meshes, the abstract result also applies to low order mixed finite elements, which are usually stable only for some special mesh structures.

Key words: Generalized Stokes equations, Mixed methods, Multigrid methods.

1. Introduction

In this paper, we consider the following generalized stationary Stokes equations:

$$\begin{cases} -\Delta \underset{\sim}{u} + \nabla p = \underset{\sim}{F}, & \text{in } \Omega, \\ \operatorname{div} \underset{\sim}{u} = G, & \text{in } \Omega, \\ \underset{\sim}{u} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded convex domain in R^2, $\underset{\sim}{u}$ represents the velocity of fluid, p its pressure; $\underset{\sim}{F}$ and G are external force and source terms. Note that the source must satisfy the compactability condition of having zero mean value, and (1.1) reduces to the stationary Stokes equations when $G \equiv 0$.

The mixed variational formulation of the generalized Stokes equations with arbitrary given force $\underset{\sim}{f}$ and source g is to find $[\underset{\sim}{u}, p] \in (H_0^1(\Omega))^2 \times L_0^2(\Omega)$ such that

$$\begin{cases} (\nabla \underset{\sim}{u}, \nabla \underset{\sim}{v}) - (p, \operatorname{div} \underset{\sim}{v}) = (\underset{\sim}{f}, \underset{\sim}{v}), & \forall \underset{\sim}{v} \in (H_0^1(\Omega))^2, \\ (q, \operatorname{div} \underset{\sim}{u}) = (g, q), & \forall\, q \in L_0^2(\Omega), \end{cases} \tag{1.2}$$

or equivalently, find $[\underset{\sim}{u}, p] \in (H_0^1(\Omega))^2 \times L_0^2(\Omega)$ such that

$$\mathcal{L}([\underset{\sim}{u}, p], [\underset{\sim}{v}, q]) = (\underset{\sim}{f}, \underset{\sim}{v}) - (g, q), \quad \forall\, [\underset{\sim}{v}, q] \in (H_0^1(\Omega))^2 \times L_0^2(\Omega), \tag{1.3.1}$$

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where $(\cdot,\cdot) \equiv (\cdot,\cdot)_\Omega$ denotes the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$, $L^2_0(\Omega)$ is the space of $L^2(\Omega)$–integrable functions which have zero mean value (cf. [7] for space notations) and

$$\mathcal{L}([\underset{\sim}{u},p],[\underset{\sim}{v},q]) = (\nabla \underset{\sim}{u}, \nabla \underset{\sim}{v}) - (p, \operatorname{div} \underset{\sim}{v}) - (q, \operatorname{div} \underset{\sim}{u}). \tag{1.3.2}$$

Note that when $\underset{\sim}{f} = \underset{\sim}{F}$ and $g = G$, (1.2) or (1.3) is the variational formulation of (1.1).

It is well–known (cf. [13] and [14]) that the problem (1.2) is uniquely solvable if $\underset{\sim}{f} \in (H^{-1}(\Omega))^2$, $g \in L^2_0(\Omega)$. Moreover, if $\underset{\sim}{f} \in (L^2(\Omega))^2$, $g \in L^2_0(\Omega) \cap H^1(\Omega)$, then the solution $[\underset{\sim}{\sigma}, \tau] \in (H^2(\Omega) \cap H^1_0(\Omega))^2 \times (H^1(\Omega) \cap L^2_0(\Omega))$ and there holds

$$\|\underset{\sim}{\sigma}\|_{H^\ell(\Omega)} + \|\tau\|_{H^{\ell-1}(\Omega)} \le C[\|\underset{\sim}{f}\|_{H^{\ell-2}(\Omega)} + \|g\|_{H^{\ell-1}(\Omega)}], \quad \ell = 1, 2. \tag{1.4}$$

To describe mixed finite element methods for the generalized Stokes equations, we begin with the triangulations of the domain Ω. Let $\mathcal{T}_k (k \ge 0)$ be a quasi–uniform triangular or rectangular partition of Ω with mesh size h_k, that is, there exists some constant $\alpha_0 > 0$, $\theta_0 > 0$ such that

$$h_K \ge \alpha_0 \rho_k, \qquad \theta_K \ge \theta_0, \quad \forall K \in \mathcal{T}_k,\ k \ge 0, \tag{A.0}$$

where h_K, θ_K and ρ_K denote, respectively, the diameter of K, the smallest angle of K and the the diameter of the largest ball contained in K. For simplicity, we also assume that $\overline{\Omega} = \cup_{K \in \mathcal{T}_k} \overline{K}$. Finally, in order to get optimal order algorithm we assume that the mesh sizes of two consecutive meshes are related as follows (cf. subsection 5.4 for the other restrictions):

$$\alpha_1^{-1} h_k \le h_{k+1} < h_k, \quad k \ge 0, \tag{1.5}$$

for some constant $\alpha_1 > 1$. Obviously, for a nested mesh family, namely, $\mathcal{T}_k$ is obtained by connecting the midpoints of the three edges of all triangles of $\mathcal{T}_{k-1}$ or by linking the midpoints of two opposite sides of all rectangles of $\mathcal{T}_{k-1}$, (1.5) holds with $\alpha_1 = 2$.

Let $\underset{\sim}{X}_k \subset (L^2(\Omega))^2, M_k \subset L^2_0(\Omega)$ be two finite element approximate spaces of $(H^1_0(\Omega))^2$ and $L^2_0(\Omega)$ associated with $\mathcal{T}_k$. The mixed finite element method for (1.2) at level k is to find $[\underset{\sim}{u}_k, p_k] \in \underset{\sim}{X}_k \times M_k$ such that

$$\begin{cases} (\nabla \underset{\sim}{u}_k, \nabla \underset{\sim}{v})_k - (p_k, \operatorname{div} \underset{\sim}{v})_k = (\underset{\sim}{f}, \underset{\sim}{v})_k, & \forall\ \underset{\sim}{v} \in \underset{\sim}{X}_k, \\ (q, \operatorname{div} \underset{\sim}{u}_k)_k = (g, q)_k, & \forall\ q \in M_k, \end{cases} \tag{1.6}$$

or equivalently, find $[\underset{\sim}{u}_k, p_k] \in \underset{\sim}{X}_k \times M_k$ such that

$$\mathcal{L}_k([\underset{\sim}{u}_k, p_k], [\underset{\sim}{v}, q]) = (\underset{\sim}{f}, \underset{\sim}{v})_k - (g, q)_k, \quad \forall\ [\underset{\sim}{v}, q] \in \underset{\sim}{X}_k \times M_k, \tag{1.7}$$

where

$$(\cdot,\cdot)_k = \sum_{K \in \mathcal{T}_k} (\cdot,\cdot)_K, \tag{1.8}$$

$$\mathcal{L}_k([\underset{\sim}{u},p],[\underset{\sim}{v},q]) = (\nabla \underset{\sim}{u}, \nabla \underset{\sim}{v})_k - (p, \operatorname{div} \underset{\sim}{v})_k - (q, \operatorname{div} \underset{\sim}{u})_k. \tag{1.9}$$

It is well–known that $\underset{\sim}{X}_k$ and M_k must satisfy the following Babuška–Brezzi condition in order to guarantee the existence and stability of the mixed finite element approximations:

$$\sup_{\underset{\sim}{v}_k \in \underset{\sim}{X}_k} \frac{|(q,\ \operatorname{div}\ \underset{\sim}{v}_k)|}{\|\underset{\sim}{v}_k\|_k} \ge \gamma_0 \|q\|_{L^2(\Omega)}, \quad \forall\ q \in M_k, \tag{1.10}$$