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LEAST-SQUARES MIXED FINITE ELEMENT METHODS FOR NONLINEAR PARABOLIC PROBLEMS*[1)]

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Abstract

Two least-squares mixed finite element schemes are formulated to solve the initial-boundary value problem of a nonlinear parabolic partial differential equation and the convergence of these schemes are analyzed.

Key words: Least-squares algorithm, Mixed finite element, Nonlinear parabolic problems, Convergence analysis.

1. Introduction

A large number of physical phenomena are modeled by partial differential equations or systems of parabolic type in an evolutionary or elliptic type at steady state. It is frequently the case that a good approximation of some function of the gradient of the solution to the differential equation (which may represent, for example, a velocity field or electric field) is at least as important as an approximation of the solution itself (which may represent, respectively, a pressure or an electric potential). Many mixed element methods compute simultaneously the solution and the gradient of the solution with the same or higher order of accuracy than the solution itself. The mixed methods were described and analyzed by many authors. It has been observed that in many cases mixed finite element methods give better approximations for the flux variable than classical Galerkin methods. However, a mixed formulation is more difficult to be handled and, in general, is more expensive from a computational point of view because it loses positive definite property. Recently, there has been an increasing interest in the applications of least-squares finite element algorithms to various problems steady or evolutionary. Many works on least-squares finite element schemes and their applications to various boundary value problems of elliptic equations or systems have been done and some systematic theories on ellipticity and error estimates have been also estabished, e.g., see [2], [3], [7]-[12], [15]-[18], [22] and [23]. In recent years, the least-squares finite element methods have been extended to time-dependent problems, e.g., see [13], [21] and [25], and several numerical results showed that least-squares finite element methods are also very effective to evolutionary problems. However, the theory on convergence of least-squares finite element methods for time-dependent problems has not been obtained.

The purpose of this paper is to analyze the least-squares mixed finite element methods for nonlinear parabolic problems written as a first-order system. Let Ω be an open bounded domain in $\mathbf{R}^d$, $d = 2, 3$, with a Lipschitz continuous boundary Γ. As a model problem, we consider the

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following initial-boundary value problem of a nonlinear parabolic equation

$$\begin{array}{ll} (a) & c(u)\frac{\partial u}{\partial t} - \sum\limits_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(u)\frac{\partial u}{\partial x_j}) = f(u), \quad \text{in } \Omega,\ 0 < t \le T; \\ (b) & u = 0, \quad \text{on } \Gamma_D; \quad \sum\limits_{i,j=1}^{d} a_{ij}(u)\frac{\partial u}{\partial x_j}\nu_i = 0, \quad \text{on } \Gamma_N; \quad 0 \le t \le T; \\ (c) & u = u_0, \quad \text{in } \Omega,\ t = 0, \end{array} \tag{1.1}$$

where the coefficients $c(v) \geq c_* > 0$ is a continuous positive functions, $\mathcal{A}(v) = (a_{ij}(v))_{d\times d}$ is a uniformly positive definite matrix function and $\nu = (\nu_1, \cdots, \nu_d)^\top$ is the unit vector normal to Γ_N. In general, the coefficients $c(v)$, $a_{i,j}(v)$ $(1 \leq i, j \leq d)$ and $f(v)$ are also dependent upon (x, t). For convenient sake and without loss of the generality, we assume that these coefficients only depend upon the unknown function.

The nonlinear parabolic problem (1.1) may be rewritten as a nonlinear first-order system of form

$$\begin{array}{ll} (a) & c(u)\frac{\partial u}{\partial t} - \mathbf{div}\sigma = f(u), \quad \text{in } \Omega,\ 0 < t \le T; \\ (b) & \sigma = \mathcal{A}(u)\nabla u, \quad \text{in } \Omega,\ 0 < t \le T; \\ (c) & u = 0, \quad \text{on } \Gamma_D; \quad \sigma\cdot\nu = 0, \quad \text{on } \Gamma_N; \quad 0 \le t \le T; \\ (d) & u = u_0, \quad \text{in } \Omega, \quad t = 0, \end{array} \tag{1.2}$$

where ∇ is the gradient operator and **div** is the divergence operator.

The paper is organized in the following way. Two least-squares mixed finite element schemes and their split parallel forms are formulated in section 2 and the theory on convergence of these schemes are established in section 3.

2. Least-Squares Mixed Element Schemes for Nonlinear Parabolic Problem

In this section, we formulate two least-squares mixed element schemes to solve (1.2). We consider a first-order mixed system equivalent to the nonlinear parabolic first-order system (1.2)

$$\begin{array}{ll} (a) & \frac{\partial u}{\partial t} - w = 0, \quad \text{in } \Omega, \quad 0 < t \le T; \\ (b) & \frac{\partial}{\partial t}(\widetilde{\mathcal{A}}(u)\sigma) - \nabla w = 0, \quad \text{in } \Omega, \quad 0 < t \le T; \\ (c) & c(u)w - \mathbf{div}\sigma = f(u), \quad \text{in } \Omega, \quad 0 < t \le T; \\ (d) & u = 0, \quad \text{on } \Gamma_D; \quad \sigma\cdot\nu = 0, \quad \text{on } \Gamma_N; \quad 0 \le t \le T; \\ (e) & u = u_0, \quad \text{in } \Omega, \quad t = 0, \end{array} \tag{2.1}$$

where $\widetilde{\mathcal{A}}$ denotes the inverse matrix of $\mathcal{A}$.

We introduce usual Sobolev spaces $\mathbf{W}^{k,p}(\Omega)$ $(k \geq 0,\ 1 \leq p \leq \infty)$ defined on Ω with usual norms $\|\cdot\|_{W^{k,p}(\Omega)}$ as in [1]. Let $\mathbf{H}^k(\Omega) = \mathbf{W}^{k,2}(\Omega)$. We define inner products in $\mathbf{L}^2(\Omega)$ and $(\mathbf{L}^2(\Omega))^d$ as follows

$$(u, v) = \int_\Omega u(x)v(x)dx, \quad \forall\ u, v \in \mathbf{L}^2(\Omega), \quad (\sigma, \omega) = \sum_{i=1}^{d}(\sigma_i, \omega_i), \quad \forall\ \sigma, \omega \in (\mathbf{L}^2(\Omega))^d;$$

and the spaces $\mathbf{H} = \{\omega \in (\mathbf{L}^2(\Omega))^d;\ \ \mathbf{div}\ \omega \in \mathbf{L}^2(\Omega),\ \ \omega\cdot\nu = 0 \text{ on } \Gamma_N.\}$, $\mathbf{S} = \{v \in \mathbf{H}^1(\Omega); v = 0 \text{ on } \Gamma_D.\}$. Let $\mathbf{T}_{h_\sigma}$ and $\mathbf{T}_{h_u}$ be two families of finite element partitions of the domain Ω, where h_σ and h_u are mesh parameters, which generally denote the largest of diameters of elements in partitions $\mathbf{T}_{h_\sigma}$ and $\mathbf{T}_{h_u}$, respectively. In practical applications, the partitions $\mathbf{T}_{h_\sigma}$ and $\mathbf{T}_{h_u}$ are same. Here, we wish to emphasize their independence in calculation and convergence analysis. Construct the finite element function spaces $\mathbf{H}_{h_\sigma} \subset \mathbf{H}$ on $\mathbf{T}_{h_\sigma}$ and $\mathbf{S}_{h_u} \subset \mathbf{S}$ on $\mathbf{T}_{h_u}$. Let