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# A PRECONDITIONER FOR COUPLING SYSTEM OF NATURAL BOUNDARY ELEMENT AND COMPOSITE GRID FINITE ELEMENT*1)

Qi-ya Hu
(*ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

De-hao Yu
(*LSEC, ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

**Abstract**

In this paper, based on the natural boundary reduction advanced by Feng and Yu, we discuss a coupling BEM with FEM for the Dirichlet exterior problems. In this method the finite element grids consist of fine grid and coarse grid so that the singularity at the corner points can be handled conveniently. In order to solve the coupling system by the preconditioning conjugate gradient method, we construct a simple preconditioner for the "stiffness" matrix. Some error estimates of the corresponding approximate solution and condition number estimate of the preconditioned matrix are also obtained.

*Key words*: Natural boundary reduction, Composite grid, Error estimate, Preconditioner, Condition number.

## 1. Introduction

The coupling of boundary elements and finite elements is of great importance for the numerical treatment of boundary value problems posed on unbounded domains. It permits us to combine the advantages of boundary elements for treating domains extended to infinity with those of finite elements in treating the complicated bounded domains.

The standard procedure of coupling the boundary element and finite element methods is described as follows. First, the (unbounded) domain is divided into two subregions, a bounded inner region and an unbounded outer one, by introducing an auxiliary common boundary. Next, the problem is reduced to an equivalent one in the bounded region. There are many ways to accomplish this reduction (refer to [2]-[6], [9]).

The natural boundary reduction method proposed by Feng and Yu [4] has obvious advantages over the usual boundary reduction methods: the coupling bilinear form preserve automatically the symmetry and coerciveness of the original bilinear form, so not only the analysis of the discrete problem is simplified, but also the optimal error estimates and the numerical stability are restored (see [4] and [14]).

It is well known that the analytic solution of the Dirichlet exterior problem is in general singular at the corner points. The fast adaptive composite grid (iteration) method advanced by

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McCormick (refer to [1], [7] and [8]) is very effective in dealing with this kind of local singularity. However, it can not be applied directly to the case of unbounded domain.

In the present paper we combine the composite grid method with the coupling method of natural boundary element and finite element to handle the corner singularity of the Dirichlet exterior problems. Under suitable assumptions we obtain the optimal error estimates of the corresponding approximate solutions. The underlying linear system is difficult to solve directly due to the complicated structure (which is neither sparse nor band). Instead, we use the preconditioning conjugate gradient (PCG) method by constructing a kind of simple preconditioner for the coupled "stiffness" matrix. We show that condition number of the preconditioned matrix is independent of the (coarse and fine) mesh sizes. Moreover, we give numerical examples to illustrate our theoretical results.

## 2. The Natural Boundary Reduction

We consider the following model exterior Dirichlet problem in two dimensions:

$$\begin{cases} -\Delta u = f & in \;\; \Omega^c = \mathbf{R}^2 \backslash (\Omega \cup \Gamma), \\ u = g & on \;\; \partial\Omega, \end{cases} \tag{2.1}$$

subject to the asymptotic conditions

$$u(x,y) = \beta + O(\frac{1}{r}) \quad as \;\; r = \sqrt{x^2+y^2} \to \infty,$$

with $\beta$ be a constant, where $\Omega$ is a Lipschitz bounded domain. Assume that the given functions $f$ and $g$ satisfy (see [6]): $supp\ f \subset \Omega_b$ and $f \in H^{-1}(\Omega_b)$ with some $\Omega_b$ being a bounded domain and containing $\Omega$; $g \in H^{\frac{1}{2}}(\partial\Omega)$.

The variational form of the boundary value problem (2.1) is: to find $u \in \bar{H}^1(\Omega^c)$, such that

$$D(u,v) = (f,v), \quad \forall\, v \in \bar{H}_0^1(\Omega^c), \tag{2.2}$$

where

$$\bar{H}^1(\Omega^c) = \{v : \frac{v}{\sqrt{(r^2+1)}\cdot \ln(r^2+2)}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\Omega^c)\},$$

$$\bar{H}_0^1(\Omega^c) = \{v : v \in \bar{H}^1(\Omega^c), v|_{\partial\Omega} = 0\}$$

and

$$D(u,v) = (\nabla u, \nabla v), \;\; \forall\, u,v \in \bar{H}^1(\Omega^c), \tag{2.3}$$

with $(\cdot,\cdot)$ be the $L^2$ innerproduct on $\Omega^c$.

Let $\Omega_0 \subset \Omega_b$ is a circle disc (with the radius $R$) containing $\Omega$. Set $\Omega_1 = \Omega^c \cap \Omega_0$ and $\Omega_2 = \Omega_0^c = \mathbf{R}^2 \backslash \Omega_0$. We assume that the ratio of the area of $\Omega_1$ over the area of $\Omega$ is not small.

Let $\Gamma$ denote the boundary of $\Omega_0$. It follows by Green formula that

$$D_1(u,v) = (f,v)_{\Omega_1} - \int_\Gamma \frac{\partial u}{\partial n} v ds, \quad \forall v \in \bar{H}_0^1(\Omega^c). \tag{2.4}$$

Let $G(p,p')$ denote the Green function of the Laplace operator on the domain $\Omega_2$, which satisfies

$$\begin{cases} -\Delta G(p,p') = \delta(p-p'), \forall\, p,p' \in \Omega_2, \\ G(p,p')|_{p\in\Gamma} = 0, \;\; \forall \;\; p' \in \Omega_2. \end{cases}$$

Set $v = G(p,p')$ in the second Green formula

$$\int\int_{\Omega_2} (v\Delta u - u\Delta v) dp' = \int_\Gamma (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}) ds,$$

we obtain (refer [13])

$$u(p) = \int\int_{\Omega_2} f\cdot G(p,p')dp' - \int_\Gamma \frac{\partial}{\partial n'} G(p,p')\cdot u(p')dp', \;\; \forall\, p \in \Omega_2.$$