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THE SOLVABILITY CONDITIONS FOR INVERSE EIGENVALUE PROBLEM OF ANTI-BISYMMETRIC MATRICES*1)

Dong-xiu Xie
(*Beijing Institute of Machinery Industry, Beijing 100085, China*)

Xi-yan Hu
(*Department of Applied Mathematics, Hunan University, Changsha 410082, China*)

Lei Zhang
(*Hunan Computing Center, Changsha 410012, China*)

Abstract

This paper is mainly concerned with solving the following two problems:

Problem I. Given $X \in C^{n\times m}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m) \in C^{m\times m}$. Find $A \in ABSR^{n\times n}$ such that

$$AX = X\Lambda$$

where $ABSR^{n\times n}$ is the set of all real $n \times n$ anti-bisymmetric matrices.

Problem II. Given $A^* \in R^{n\times n}$. Find $\hat{A} \in S_E$ such that

$$\|A^* - \hat{A}\|_F = \min_{A\in S_E} \|A^* - A\|_F,$$

where $\|\cdot\|_F$ is Frobenius norm, and S_E denotes the solution set of Problem I.

The necessary and sufficient conditions for the solvability of Problem I have been studied. The general form of S_E has been given. For Problem II the expression of the solution has been provided.

Key words: Eigenvalue problem, Norm, Approximate solution.

1. Introduction

Inverse eigenvalue problem has widely been used in engineering. For example inverse eigenvalue method is a useful means in vibration design and vibration control of flyer. In recent years a serial of good conclussions have been made for inverse eigenvalue problem. However, inverse problems of anti-bisymmetric matrices have not be concerned yet. In this paper we will discuss this problem.

We denote the complex $n \times m$ matrix space by $C^{n\times m}$, the real $n \times m$ matrix space by $R^{n\times m}$, and $R^n = R^{n\times 1}$, the set of all matrices in $R^{n\times m}$ with rank r by $R_r^{n\times m}$, the set of all $n\times n$ orthogonal matrices by $OR^{n\times n}$, the set of all $n\times n$ anti-symmetric matrices by $ASR^{n\times n}$, the column space, the null space and the Moore–Penrose generalized inverse of a matrix A by $R(A), N(A), A^+$ respectively, the identity matrix of order n by I_n, the Frobenius norm of A by $\|A\|_F$. We define inner product in space $R^{n\times m}$, $(A, B) = \mathrm{tr}(B^T A) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ij}, \quad \forall A, B \in R^{n\times m}$. Then $R^{n\times m}$ is a Hilbert inner product space. The norm of a matrix produced by the

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inner product is Frobenius norm. Let $S_k = (e_k, e_{k-1}, \cdots, e_1) \in R^{k\times k}$ in which e_i is the i-th Cloumn of the identity matrix I_k.

Definition 1. $A = (a_{ij}) \in R^{n\times n}$, *if*

$$a_{ij} = -a_{ji}, \qquad a_{ij} = -a_{n-j+1,n-i+1}, \qquad i,j = 1,2,\ldots,n$$

then A is called a anti-bisymmetric matrix. The set of all anti-bisymmetric matrices is denoted by $ABSR^{n\times n}$.

Now we consider the following problems:

Problem I. Given $X \in C^{n\times m}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m)$. Find $A \in ABSR^{n\times n}$ such that

$$AX = X\Lambda.$$

Problem II. Given $A^* \in R^{n\times n}$. Find $\hat{A} \in S_E$ such that

$$\|A^* - \hat{A}\|_F = \min_{A\in S_E} \|A^* - A\|_F,$$

where S_E is the solution set of problem I.

At first, in this paper, we will discuss the character of eigenvector for anti-bisymmetric matrices. Then we will give the necessary and sufficient conditions for the solvability of Problem I and the expression of the general solution of Problem I in real number field, and point out S_E is a closed convex set. At last, we will prove that there exists a unique solution of Problem II and give an expression of the solution for Problem II.

2. The Solvability Conditions and General Form of the Solutions for Problem I in Real Number Field

At first we discuss the construction of $ABSR^{n\times n}$ and the character of eigenvector for matrices in $ABSR^{n\times n}$.

Let

$$k = [\frac{n}{2}], \qquad \text{[x] is integer number that is not greater than x.} \tag{2.1}$$

When $n = 2k$,

$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} I_k & I_k \\ S_k & -S_k \end{pmatrix}, \quad D^T D = I_n; \tag{2.2}$$

and when $n = 2k+1$,

$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{pmatrix}, \qquad D^T D = I_n. \tag{2.3}$$

Lemma 1. $A \in ABSR^{n\times n}$ *if and only if*

$$A = S_n A S_n, \qquad A = -A^T$$

Theorem 1.

$$ABSR^{2k\times 2k} = \left\{ \begin{pmatrix} M & HS_k \\ S_k H & S_k M S_k \end{pmatrix} \middle| \ M, H \in ASR^{k\times k} \right\}. \tag{2.4}$$

$$ABSR^{(2k+1)\times(2k+1)} = \left\{ \begin{pmatrix} N & C & HS_k \\ -C^T & 0 & -C^T S_k \\ S_k H & S_k C & S_k N S_k \end{pmatrix} \middle| \ N, H \in ASR^{k\times k}, C \in R^k \right\}. \tag{2.5}$$