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SUPERAPPROXIMATION PROPERTIES OF THE INTERPOLATION OPERATOR OF PROJECTION TYPE AND APPLICATIONS*1)

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Abstract

Some superapproximation and ultra-approximation properties in function, gradient and two-order derivative approximations are shown for the interpolation operator of projection type on two-dimensional domain. Then, we consider the *Ritz* projection and *Ritz-Volterra* projection on finite element spaces, and by means of the superapproximation elementary estimates and *Green* function methods, derive the superconvergence and ultraconvergence error estimates for both projections, which are also the finite element approximation solutions of the elliptic problems and the Sobolev equations, respectively.

Key words: Interpolation operator of projection type, Finite element, Superconvergence.

1. Introduction

Finite element superconvergence property has long attracted considerable attentions since its practical importance in enhancing the accuracy of finite element calculation and in adaptive computing via a posteriori error estimate. In this field affluent research results have been achieved. Some recent developments include the patch recovery technique by Zienkiewicz and Zhu [1], the computer-based approach by Babuska, Strouboulis, et al. [2], superconvergence via local symmetry by Schatz, Sloan and Wahlbin [3], and the integral identity method by Lin and his colleagues [4-5], etc. For a more complete literature on superconvergence, the reader is referred to Wahlbin's book [6], Chen and Huang's book [7], and a recent conference proceeding edited by Krizek et al. [8].

In article [4,9], Lin Qun presented a new type of interpolation operator into the finite element spaces, ie. the interpolation operator of projection type, and remarked that it will approximate the finite element solutions much better than the usual *Lagrange* interpolation. Thus, the interpolation operator of projection type provides a new powerful means in the research of finite element superconvergence. In this paper, we first investigate the interpolation operator of projection type in two-dimensional rectangular domain case and find out many delicate superapproximation and ultra-approximation properties in function, gradient and two-order derivative approximations, some of them are unknown for the *Lagrange* interpolation operator, where by ultra-approximation we mean that the approximate order is two-order higher than the global optimal approximate order. Then, we consider the *Ritz* projection on the tensor-product

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finite element spaces associated with the general partial differential operator of second order

$$A = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} a_{ij} \frac{\partial}{\partial x_i} + \sum_{i=1}^{2} a_i \frac{\partial}{\partial x_i} + a_0 I$$

Utilizing the superapproximation properties of interpolation operator of projection type and the *Green* function methods, we prove that the *Lobatto*, *Gauss* and *quasi-Lobatto* points on each subdivision element are superconvergence points of *Ritz* projection in function, gradient and two-order derivative approximations, successively. Furthermore, in the case of $A = -\Delta + a_0 I$, the ultraconvergence results are obtained successively for the function approximation at mesh nodal points, gradient approximation under arithmetic mean of two *Gauss* points and two-order mixed derivative approximation at *Gauss* points. Generally speaking, the finite element solution itself does not share the global superconvergence, but, in this paper, we obtain superconvergence on whole domain for the two-order mixed derivative approximation, this is a somewhat remarkable result. Finally, we study the *Ritz-Volterra* projection on finite element spaces which has been widely used in the finite element analysis for various evolution equations [11−13], and prove that those superconvergence and ultraconvergence properties shared by *Ritz* projection also hold for the *Ritz-Volterra* projection. A direct application of this projection here is the analysis of semidiscret finite element approximation to the *Sobolev* equations.

We would like to indicate that some superconvergence results concerning the *Ritz* projection in our paper may be not new[6,7,10], but they are derived by a different approach and for more general case (see operator A).

In Section 2 we introduce the interpolation operator of projection type and investigate its various superapproximation properties. Section 3 and 4 are devoted to the superconvergence of *Ritz* projection and *Ritz-Volterra* projection, respectively, and finally the semidiscret finite element approximation to the initial-boundary value problems of *Sobolev* equations is discussed.

In this article we shall use letter C to denote a generic constant which may not be the same in each occurrence and also use the standard notions for the *Sobolev* spaces and norms.

2. Interpolation Operator of Projection Type and Its Superapproximation Properties

Let element $e = e_1 \times e_2 = (x_e - h_e, x_e + h_e) \times (y_e - k_e, y_e + k_e)$, $\{L_j(x)\}_{j=0}^{\infty}$ and $\{\widetilde{L}_j(y)\}_{j=0}^{\infty}$ be the normalized orthogonal *Legendre* polynomial systems on $L_2(e_1)$ and $L_2(e_2)$, respectively. Set

$$\omega_0(x) = \widetilde{\omega}_0(y) = 1, \quad \omega_{j+1}(x) = \int_{x_e - h_e}^{x} L_j(x)\,dx, \ \ \widetilde{\omega}_{j+1}(y) = \int_{y_e - k_e}^{y} L_j(y)\,dy, \ \ j \geq 0$$

It is well known that polynomials $\omega_{k+1}(x), L_k(x)$ and $L_k'(x)$ $(k \geq 1)$ have successively $k+1$, k and $k-1$ zeros on element $e_1 = (x_e - h_e, x_e + h_e)$, and these zeros are symmetrically distributed with respect to the middle point x_e. Denote the three kinds of zero point set successively as *Lobatto*, *Gauss* and *quasi-Lobatto* point set by $N_k^{(s)} = \{g_j^{(s)}\}$, $s = 0, 1, 2$. Moreover, these polynomials also possess the following symmetry and antisymmetry

$$\omega_{2j}(x_e + x) = \omega_{2j}(x_e - x), \quad \omega_{2j-1}(x_e + x) = -\omega_{2j-1}(x_e - x) \tag{1}$$

$$L_{2j}(x_e + x) = L_{2j}(x_e - x), \quad L_{2j-1}(x_e + x) = -L_{2j-1}(x_e - x) \tag{2}$$

$$L_{2j}'(x_e + x) = -L_{2j}'(x_e - x), \quad L_{2j-1}'(x_e + x) = L_{2j-1}'(x_e - x) \tag{3}$$