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CONVERGENCE RESULTS OF RUNGE-KUTTA METHODS FOR MULTIPLY-STIFF SINGULAR PERTURBATION PROBLEMS*[1)]

Ai-guo Xiao
(*Department of Mathematics, Xiangtan University, Xiangtan 411105, China*)

Abstract

The main purpose of this paper is to present some convergence results for algebraically stable Runge-Kutta methods applied to some classes of one- and two-parameter multiply-stiff singular perturbation problems whose stiffness is caused by small parameters and some other factors. A numerical example confirms our results.

Key words: Singular perturbation problems, Runge-Kutta methods, Convergence, Multiple-stiffness.

1. Introduction

The initial value problems of ordinary differential equations in singular perturbation form often arise in many practical applications, such as chemical kinetics, automatic control et.al.(cf. [1, 7, 13-15]). The asymptotic behaviours and expansion solutions of these problems have been studied in detail by many authors(such as [1, 7, 13-15]). The initial value problems in singular perturbation form may be considered as a special class of stiff initial value problems. But it is sorry that they can't be satisfactorily covered by B-theory (cf. [3-6, 8, 11, 19]) because of their very special structures. In the recent more than ten years, many authors[8-10, 12, 16-18] have presented many important and interesting convergence results for linear multistep methods, Runge-Kutta methods, Rosenbrock methods, partitioned linearly implicit Runge-Kutta methods and general linear methods etc. applied to one-parameter singular perturbation problems(SPPs). But all these results are within the limits of the SPPs whose the right-side functions satisfy Lipschitz conditions with moderately-sized Lipschitz constants as the essential problem-characterizing parameters, we thus call these problems singly-stiff singular perturbation problems(SSPPs) because their stiffness is only caused by small parameters. For the SPPs whose stiffness is caused by small parameters and some other factors, we call them multiply- stiff singular perturbation problems(MSPPs), and the corresponding reduced problems are called stiff differential-algebraic equations(SDAEs). Some practical examples of MSPPs have been given in [21].

So far, there exists some convergence results of partitioned Runge-Kutta methods, one-leg methods and linear multistep methods for MSPPs (cf. [20, 21]). In the present paper, we will obtain some convergence results of algebraically stable Runge-Kutta methods(RKMs) for SDAEs and one-parameter MSPPs in Section 2. We will extend the results given in Section 2 to two classes of MSPPs with two parameters in Section 3. In Section 4, we will also give the convergence results of Runge-Kutta methods applied to a class of MSPPs with an algebraic constraint. In Section 5, a numerical example confirms our results.

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2. One-parameter MSPPs

Consider the SPP with one parameter

$$\begin{cases} x'(t) &= f(x,y), \qquad t\in[t_0,t_e], \\ \epsilon y'(t) &= g(x,y), \qquad 0<\epsilon<<1 \end{cases} \tag{2.1}$$

with initial values $(x(t_0),y(t_0))\in\check{G}$ admitting a smooth solution (x(t),y(t))(i.e. all derivatives of $x(t)$ and $y(t)$ up to a sufficiently high order are bounded independently of the stiffness of the problem), where $\check{G}$ is an appropriate region on $R^M\times R^N$, and the maps $f:\ \check{G}\to R^M$ and $g:\ \check{G}\to R^N$ are sufficiently smooth and satisfy

$$< f(x_1,y)-f(x_2,y), x_1-x_2 > \le m\|x_1-x_2\|^2, \quad \forall\ (x_1,y),(x_2,y)\in\check{G}, \tag{2.2a}$$

$$< g(x,y_1)-g(x,y_2), y_1-y_2 > \le -\|y_1-y_2\|^2, \quad \forall\ (x,y_1),(x,y_2)\in\check{G}, \tag{2.2b}$$

$$\|f(x,y_1)-f(x,y_2)\| \le L_1\|y_1-y_2\|, \quad \forall\ (x,y_1),(x,y_2)\in\check{G}, \tag{2.2c}$$

$$\|g(x_1,y)-g(x_2,y)\| \le L_2\|x_1-x_2\|, \quad \forall\ (x_1,y),(x_2,y)\in\check{G} \tag{2.2d}$$

with moderately-sized constants m, L_1 and L_2, where, throughout this paper, $<.,.>$ is the standard inner product in real Euclid space with the corresponding norm $\|\ .\ \|$, the matrix norm used in the following text is subject to $\|\ .\ \|$, and $\mu(.)$ denotes the logarithmic norm with respect to $<.,.>$. In the proof of the following results, we often make use of the following fact

$$\mu(f_x+\Phi)\le m+L, \quad for\ \|\ \Psi\ \|\le L, \tag{2.3}$$

where $\Psi\in R^{M\times M}$, L is moderately-sized.

A Runge-Kutta method (A,b,c) with

$$A=[a_{ij}]\in R^{s\times s}, \quad b^T=(b_1,b_2,\cdots,b_s), \quad c^T=(c_1,c_2,\cdots,c_s)$$

applied to the problem (2.1) reads

$$X_i = x_n + h\sum_{j=1}^{s} a_{ij} f(X_j,Y_j), \quad i=1,2,\cdots,s, \tag{2.4a}$$

$$\epsilon Y_i = \epsilon y_n + h\sum_{j=1}^{s} a_{ij} g(X_j,Y_j), \quad i=1,2,\cdots,s, \tag{2.4b}$$

$$x_{n+1} = x_n + h\sum_{i=1}^{s} b_i f(X_i,Y_i), \tag{2.4c}$$

$$\epsilon y_{n+1} = \epsilon y_n + h\sum_{i=1}^{s} b_i g(X_i,Y_i), \tag{2.4d}$$

with the starting values x_0 and y_0, where $h>0$ is the stepsize, x_n, y_n, X_i and Y_i are approximations to the exact solutions $x(t_n), y(t_n), x(t_n+c_ih)$ and $y(t_n+c_ih)$ respectively, and $n=0,1,\cdots,\check{N}$, $(\check{N}+1)h\le t_e-t_0$.