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ON BANDLIMITED SCALING FUNCTION*1)

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Abstract

This paper discuss band–limited scaling function, especially on the interval band case and three interval bands case, its relationship to oversampling property and weakly translation invariance are also studied. At the end, we propose an open problem.

Key words: Scaling function, Oversampling property, Weakly translation invariance, Aliasing error.

1. Introduction

A MRA of $L^2(R)$ is an increasing family of subspaces $\{V_m\}_{m\in Z}$, with
1) $V_m \subset V_{m+1}$,
2) $\underline{f(x) \in} V_m$ if and only if $f(2x) \in V_{m+1}$,
3) $\bigcup_m V_m = L^2(R)$ and $\bigcap_m V_m = \{0\}$,
4) There exists a function $\varphi(x) \in V_0$(called **scaling function** of MRA $\{V_m\}_m$) such that $\{\varphi(x-m)\}_m$ forms a Riesz basis of V_0.

An **orthogonal MRA** is a MRA with $\{\varphi(x-m)\}_m$ forming an orthogonal Riesz basis of V_0. Sometimes, φ (or a MRA $\{V_m\}_m$) is said to be **orthonormal** if $\{\varphi(x-m)\}_m$ is an orthonormal Riesz basis of V_0.

Clearly, $\{\varphi(2^j x-m)\}_{m\in Z}$forms the basis of V_j. Let $W_0 = V_1 \ominus V_0$ be the direct complement of V_0 in V_1, $\psi(x) = \sum d_k \varphi(2x-k)$ (called wavelet) such that $\{\psi(x-n)\}_{n\in Z}$ forms a Riesz basis of W_0. $\varphi(x) = \sum c_k \varphi(2x-k)$, then we have

$$\widehat{\varphi}(\omega) = m_0(\omega/2)\widehat{\varphi}(\omega/2), \tag{1.1}$$

and

$$\widehat{\psi}(\omega) = m_1(\omega/2)\widehat{\varphi}(\omega/2), \tag{1.2}$$

where $m_0(\omega) = \frac{1}{2}\sum_k c_k e^{-ik\omega}, m_1(\omega) = \frac{1}{2}\sum_k d_k e^{-ik\omega}$ and the Fourier transform is defined by

$$\widehat{f}(\omega) = \int_R f(t)e^{-i\omega t}dt \quad \text{for} \quad f(t) \in L^2(R) \cap L^1(R). \tag{1.3}$$

From (1.1) and (1.2) we know that the scaling function play a very important role for constructing a wavelet $\psi(x)$. How to find a mild wavelet for some problems can be reduced to find a proper scaling function $\varphi(x)$.

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As a special important case, [4] and [7] have dealed with the compactly supported scaling function systems, it is to solve the following dilation equation with finite non–zero c_k

$$\varphi(x) = \sum c_k \varphi(2x-k). \tag{1.4}$$

However, many important scaling function, such as Shannon Wavelet and Meyer Wavelet [9] are not compactly supported but band limited, therefore dealing with band–limited scaling function in more general case is necessary and useful. Unfortunately, this problem is not so easy to solve as the compactly support case. Up to now, there are yet no systematic results about the problem, except some sporadic works such as [3].

In the paper, we focus on the scaling function with interval band and finite unions of interval bands. After general discussing on scaling function In section 3, a characterization of bandlimited scaling function will be given which formerly only can hold with translation invariance (see [9] or section 3). By the way, we also prove interval band scaling functions are translation invariance and have oversampling property (see [12] or section 3). Additionally, the aliasing error with oversampling property (see [J] or section 1) is estimated.

In section 4, we will discuss the scaling function with three interval bands and follow some similar results as section 3. At the end of the paper, we propose an open problem.

2. Elementary Properties of Scaling Function

The scaling functions of a MRA have many important elementary properties. We here as completion will list some propositions concerning these properties which will be applied to the following sections. Some proofs that appeared in other references will be omitted.

Proposition 2.1. *$\{V_j\}$ is an orthonormal MRA with scaling function $\varphi(t)$, $\widehat{\varphi}(\xi)$ is continuous, then*

(a) $\sum_k |\widehat{\varphi}(\omega+2k\pi)|^2 = 1, |m_0(\omega)|^2 + |m_1(\omega)|^2 = 1 \quad a,e \quad \omega \in R,$

(b) $\widehat{\varphi}(0) = 1$ *and* $\widehat{\varphi}(2k\pi) = 0 \quad$ *for* $\quad k \neq 0,$

(c) $\sum_k \varphi(x-k) = \widehat{\varphi}(0)$ *when* $\varphi(x) \in L^1(R)$.

Proof of (a), (b) and (c) are well known and can be found in many books concerning wavelet, such as [6], [8], [5] and [11].

Proposition 2.2. *$\varphi(x)$ is an orthonormal scaling function of MRA $\{V_m\}$, supp$\widehat{\varphi}(\omega) = \Omega$, then*

(a) $\Omega \subset 2\Omega,$

(b) $\bigcup_k \{\Omega + 2k\pi\} = R,$

(c) $\bigcup_j \{2^j\Omega\} = R.$

Proof. The proof is easy and also can be found in [9].

(a) follows from $\Omega = \text{supp}\widehat{\varphi}(\omega) \subset \text{supp}\widehat{\varphi}(\frac{\omega}{2}) = 2\Omega$. due to (1.1).

(b) follows from (a) of proposition 2.1.

(c) follows from $\overline{\bigcup_m V_m} = L^2(R)$.

Proposition 2.3. *$\varphi(t)$ is an orthonormal scaling function with supp$\widehat{\varphi}(\omega) = \Omega$, then $|\Omega| \geq 2\pi$ where $|\Omega|$ is the Lebesque measure of measurable set Ω.*

Proof. Let $R = \sum_k [2k\pi, 2(k+1)\pi)$, then $\Omega = \sum_k \Omega_k$, where $\Omega_k = \Omega \bigcap [2k\pi, 2(k+1)\pi)$. Clearly, $\bigcup_k \{\Omega_k - 2k\pi\} \subset [0, 2\pi]$. We can claim

$$|[0,2\pi] \setminus \bigcup_k \{\Omega_k - 2k\pi\}| = 0. \tag{2.1}$$