Journal of Computational Mathematics, Vol.20, No.4, 2002, 413–428.

LAGUERRE PSEUDOSPECTRAL METHOD FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS[*1)]

Cheng-long Xu
(*Department of Mathematics, Tongji University, Shanghai 200092, China*)

Ben-yu Guo
(*Department of Mathematics, Shanghai Normal University, Shanghai 200234, China*)

Abstract

The Laguerre Gauss-Radau interpolation is investigated. Some approximation results are obtained. As an example, the Laguerre pseudospectral scheme is constructed for the BBM equation. The stability and the convergence of proposed scheme are proved. The numerical results show the high accuracy of this approch.

Key words: Laguerre pseudospectral method, Nonlinear differential equations.

1. Introduction

In scientific computations, we often need to solve differential equations in unbounded domains numerically, e.g., see Gottlieb and Orszag [1], Canuto, Hussaini, Quarteroni and Zang [2], Bernardi and Maday [3], and Guo [4]. Usually we set up some artificial boundaries, impose certain artificial boundary conditions and then resolve them. Whereas these treatments cause additional errors. One of reasonable ways for solving such problems is to use spectral method or pseudospectral method related to orthogonal systems of polynomials in unbounded domains. In particular, the Laguerre spectral method and Laguerre pseudospectral method are applicable to problems on the half line. Maday, Bernaud-Thomas and Vandeven [5] established some results on the Laguerre approximation. We also refer to Funaro [6]. On the other hand, Mavriplis [7], Coulaud, Funaro and Kivian [8], and Iranzo and Falqués [9] proposed various algorithms based on the Laguerre approxiamtion. Recently Guo and Shen [10] derived some new approximation results on the Laguerre approximation, constructed some Laguerre spectral schemes for nonlinear problems, proved the stability and the convergence of proposed schemes, and obtained accurate numerical results. But in actual computations, the Laguerre pseudospectral method is more preferable, since it does not need quadratures on the half line, and so saves a lot of work and avoids the corresponding numerical errors. In addition, it is easier to deal with nonlinear terms, e.g., see Coulaud, Funaro and Kivian [8], and Iranzo and Falqués [9]. It is noted that Mastroianni and Monegato [11], also established another kind of approximation results on the generalized Laguerre interpolation and used them for some integral equations successfully.

The aim of this paper is to develop the Laguerre pseudospectral method and its applications to nonlinear partial differential equations. In the next section, we introduce certain spaces, establish some weighted imbedding inequalities, inverse inequalities and approximation results on the Laguerre -Gauss-Radau interpolation. These results play important roles in numerical analysis of the Laguerre pseudospectral method for nonlinear partial differential equations. In section 3, we take the Benjamin-Bona-Mahony (BBM) equation as an example to show how to construct reasonable Laguerre pseudospectral schems for nonlinear problems. The stability and

[*] Received January 10, 2000.
[1)] The work of the second author is partially supported by The Special Funds for Major State Basic Research Projects of China N.G1999032804 and Shanghai Natural Science Foundation N.00JC14057.

the convergence of proposed scheme are proved. The numerical results show the high accuracy of this approach. The main idea and the techniques used in this paper are also applicable to other nonlinear problems.

2. The Laguerre-Gauss-Radau Interpolation

Let $\Lambda = \{x \mid 0 < x < \infty\}$, $\bar{\Lambda} = \Lambda \cup \{0\}$ and $\omega(x) = e^{-x}$. For $1 \le p \le \infty$,

$$L^p_\omega(\Lambda) = \{\ v \mid v \ \text{ is measurable and } \ \|v\|_{L^p_\omega} < \infty\ \}$$

where

$$\|v\|_{L^p_\omega} = \begin{cases} \left(\int_\Lambda |v(x)|^p \omega(x) dx\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Lambda} |v(x)|, & p = \infty. \end{cases}$$

In particular, $L^2_\omega(\Lambda)$ is a Hilbert space equipped with the following inner product and norm

$$(u,v)_\omega = \int_\Lambda u(x)v(x)\omega(x)dx. \quad \|v\|_\omega = (v,v)^{\frac{1}{2}}_\omega.$$

For simplicity, let $\partial_x v(x) = \frac{\partial v}{\partial x}(x)$, etc.. For any non-negative integer m,

$$H^m_\omega(\Lambda) = \{\ v \mid \partial^k_x v \in L^2_\omega(\Lambda), 0 \le k \le m\}$$

equipped with the following inner product, semi-norm and norm

$$(u,v)_{m,\omega} = \sum_{k=0}^{m} (\partial^k_x u, \partial^k_x v)_\omega,$$

$$|v|_{m,\omega} = \|\partial^m_x v\|_\omega, \qquad \|v\|_{m,\omega} = (v,v)^{\frac{1}{2}}_{m,\omega}.$$

For any real $r > 0$, the space $H^r_\omega(\Lambda)$, its semi-norm $|v|_{r,\omega}$ and norm $\|v\|_{r,\omega}$ are defined by space interpolation as in Adams [12]. Furthermore

$$H^1_{0,\omega}(\Lambda) = \{\ v \mid v \in H^1_\omega(\Lambda) \text{ and } v(0) = 0\}.$$

In addition, $\|v\|_{L^\infty}$ stands for $\|v\|_{L^\infty(\Lambda)}$. We have the following imbedding inequalities.

Lemma 2.1. *For any $v \in H^1_{0,\omega}(\Lambda)$,*

$$\|e^{-\frac{x}{2}} v\|_{L^\infty} \le \sqrt{2}\|v\|^{\frac{1}{2}}_\omega |v|^{\frac{1}{2}}_{1,\omega}, \tag{2.1}$$

$$\|v\|_\omega \le 2|v|_{1,\omega}. \tag{2.2}$$

Moreover for any $v \in H^1_\omega(\Lambda)$,

$$\|e^{-\frac{x}{2}} v\|_{L^\infty} \le \sqrt{2}\|v\|_{1,\omega}. \tag{2.3}$$

Proof. (2.1) and (2.2) are proved in Guo and Shen [10]. Next, for any $x \in \Lambda$,

$$\begin{aligned} e^{-x}v^2(x) &= -\int_x^\infty \partial_y(e^{-y}v^2(y))dy = \int_x^\infty e^{-y}v^2(y)dy - 2\int_x^\infty e^{-y}v(y)\partial_y v(y)dy \\ &\le \|v\|^2_\omega + 2\|v\|_\omega |v|_{1,\omega} \le 2\|v\|^2_{1,\omega}. \end{aligned}$$

We next recall some properties of the Laguerre polynomials. The Laguerre polynomial of degree l is defined by

$$\mathcal{L}_l(x) = \frac{1}{l!} e^x \partial^l_x (x^l e^{-x}).$$

It is the l-th eigenfunction of the singular Strum-Liouville problem

$$\partial_x(xe^{-x}\partial_x v(x)) + \lambda e^{-x} v(x) = 0,$$