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ON THE FINITE VOLUME ELEMENT VERSION OF RITZ-VOLTERRA PROJECTION AND APPLICATIONS TO RELATED EQUATIONS*

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Abstract

In this paper, we present a general error analysis framework for the finite volume element (FVE) approximation to the Ritz-Volterra projection, the Sobolev equations and parabolic integro-differential equations. The main idea in our paper is to consider the FVE methods as perturbations of standard finite element methods which enables us to derive the optimal L_2 and H^1 norm error estimates, and the L_∞ and W^1_∞ norm error estimates by means of the time dependent Green functions. Our disc ussions also include elliptic and parabolic problems as the special cases.

Key words: Finite volume element, Ritz-Volterra projection, Integro-differential equations, Error analysis.

1. Introduction

Consider the integro-differential equation of Volterra type

$$\begin{array}{rll} V(t)u \equiv A(t)u + \int_0^t B(t,\tau)u(\tau)\,d\tau & = f(t), & in\ J\times\Omega \\ u(t,x) & = 0, & on\ J\times\partial\Omega \end{array} \tag{1}$$

Where Ω is a bounded convex polygon in R^2 with a boundary $\partial\Omega$, $J=(0,T]$, $T>0$, $A(t)$ is a symmetric and positive definite linear partial differential operator of second order, and $B(t,\tau)$ an arbitrary second order linear partial differential operator, both with coefficients depending smoothly on x, t, and τ for the latter. When $f(t)\in L_\infty(J;L_p(\Omega))$, problem (1) admits a unique solution $u(t)\in L_\infty(J;W_p^2(\Omega)\bigcap H_0^1(\Omega))$ and satisfies[1]

$$\|u(t)\|_{2,p} \le C(\|f(t)\|_{0,p} + \int_0^t \|f(t)\|_{0,p}d\tau\,),\ 1<p<p_0,\, t\in J \tag{2}$$

Where $p_0 = 2+\alpha$, $\alpha>0$ is a positive constant depending on the maximal inner angle of Ω, and when $\partial\Omega$ is smooth enough, $p_0=\infty$.

The so called finite element Ritz-Volterra projection[2] just is the finite element approximation $u_h(t)$ of the exact solution $u(t)$ of problem (1). Obviously, Ritz-Volterra projection is a natural generalization of the finite element Ritz projection, when $B(t,\tau)\equiv 0$, both are identical. In recent years, Ritz-Volterra projection has attracted considerable attentions since that it provides a unified and powerful means in studying the Galerkin finite element methods for many evolution equations such as parabolic and hyperbolic integro-differential equations, Sobolev equations and visco-elasticity equations, etc. [2−7] In this paper, we will investigate the finite volume element(FVE) version of the finite element Ritz-Volterra projection, that is the

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FVE approximation to the exact solution of problem (1). We will present a general error analysis framework for the FVE methods of the integro-differential equations and related equations, and establish some optimal error estimates under L_2, H^1, L_∞ and W_∞^1 norms.

FVE methods for the elliptic boundary value problems have a long history just like finite element methods. In early literatures [8,9], a so called integral finite difference methods were systematically investigated, most of the results were given in one-dimensional cases. FVE methods have also been termed as box scheme, generalized finite difference schemes or integral type difference schemes [10]. Generally speaking, FVE methods are numerical techniques lie somewhere between finite difference and finite element methods. They have a flexibility similar to that of finite element methods for handing complicated solution domain geometry and boundary conditions, and have a comparable simplicity for implementation like finite difference methods when the triangulation has simple structures. More importantly, numerical solutions generated by FVE methods usually have certain conservation features which are very desirable in many applications. However, the analysis for FVE methods is far behind that for the finite element and finite difference methods. The readers are referred to articles [10-22] for some recent developments.

Many early publications can be found on the FVE methods using linear finite elements and the related optimal H^1 norm error estimates, and some superconvergence in the discrete H^1 norms. Later the authors of [10] obtained L_2 norm error estimate of the following form:

$$\|u - u_h\| \leq Ch^2 \|u\|_{3,p}, \ p > 1 \tag{3}$$

Note that the order in this estimate is optimal, but its regularity requirement on the exact solution is not. In article [16,17], a framework based on functional analysis was presented to analyze the FVE methods, but they did not provide the optimal L_2 error estimate. The authors of [18] obtained some new error estimates by extending the techniques of [10]. In these articles, optimal H^1 and W_∞^1 error estimates and superconvergence in H^1 and W_∞^1 norms are obtained.

Recently, the authors of [23] present the L_2 error estimate of the following form:

$$\|u - u_h\| \leq C(h^2 \|u\|_2 + h^{1+\beta} \|f\|_\beta), \ 0 \leq \beta \leq 1 \tag{4}$$

It seems to be a better result compared with that in (3), because, except for the solution domain with a boundary smooth enough, the H^1 regularity on the source term does not automatically imply the H^3 regularity of the exact solution. Moreover, they also indicate, by a counter example, that the regularity requirement on the source term can not be reduced in order to obtain the optimal order error estimate.

These results just mentioned are mainly for the elliptic and parabolic problems. To our knowledge, up to now, there are no or few publications concerning the FVE methods for the integro-differential equations as above. In this paper, we will investigate the FVE methods using linear finite element for problem (1), and the Sobolev equations and parabolic integro-differential equations. The main idea in our paper is to consider FVE methods as perturbations of Galerkin finite element methods. This approach simplifies tremendously our analysis and allows us to employ the standard error analysis techniques developed for finite element methods to derive the optimal L_2 and H^1 norm error estimates. Moreover, by means of the time dependent Green function methods introduced in articles [1,6], we also obtain the optimal L_∞ and W_∞^1 norm error estimates. In our discussion below, for simplicity, we assume the operators $A(t)$ and $B(t,\tau)$ as follows

$$A(t) = -\nabla \cdot A\, \nabla \ \ ; \quad B(t,\tau) = \nabla \cdot B\, \nabla$$

Where $A = (a_{ij}(t,x))$ is a 2×2 symmetric and positive definite matrix uniformly in $J \times \Omega$, and $B = (b_{ij}(t,\tau,x))$ an arbitrary 2×2 matrix in $J \times J \times \Omega$. The results of this paper can be extended easily to cover more general models without any additional difficulties.

This paper is organized as follows. In Section 2, we formulate the FVE methods in linear finite element spaces defined on a triangulation. Section 3 is devoted to the L_2 and H^1 norm error