NONLINEAR STABILITY OF NATURAL RUNGE-KUTTA METHODS FOR NEUTRAL DELAY DIFFERENTIAL EQUATIONS*1)

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Abstract

This paper first presents the stability analysis of theoretical solutions for a class of nonlinear neutral delay-differential equations (NDDEs). Then the numerical analogous results, of the natural Runge-Kutta (NRK) methods for the same class of nonlinear NDDEs, are given. In particular, it is shown that the $(k, l)$-algebraic stability of an RK method for ODEs implies the generalized asymptotic stability and global stability of the induced NRK method.

Key words Nonlinear stability, Neutral delay differential equations, Natural Runge-Kutta methods.

1. Introduction

In the last several decades, there has been a growing interest in the numerical stability for DDEs(cf. [1-14]). In 1988, A.Bellen, Z. Jackiewicz and M.Zennaro[7] first extend the researches to the scalar linear NDDEs. Latterly, a lot of works for the systems of linear NDDEs were presented(cf.[8-12]). However, there are much difficulties to assess the numerical stability of nonlinear NDDEs. In view of this, T.Koto [13] adapted NRK methods (cf.[14]) to a class of nonlinear NDDEs in real space $\mathbb{R}^d$, and studied their asymptotic stability with a discrete analogue of the Liapunov functional.

In this paper, by an alternative approach, we further deal with the stability of theoretical and numerical solutions for a class of nonlinear NDDEs in complex space $\mathbb{C}^d$. Particularly, it is shown that a NRK method induced by a $(k, l)$-algebraically stable RK methods for ODEs, under suitable conditions, preserves the analogous stability of the original equations.

2. Test Problem and Its Stability

For giving subsequent analysis, we first set some notational conventions. Let $\langle \bullet, \bullet \rangle, \| \bullet \|$ denote the inner product and the induced norm in space $\mathbb{C}^d$, respectively. Correspondingly, the inner product and the induced norm in space $(\mathbb{C}^d)^l$ are defined as follows:

$$\langle U, V \rangle = \sum_{i=1}^{l} \langle u_i, v_i \rangle, \quad \| U \|^2 = \langle U, U \rangle,$$

where $U = (u_1, u_2, \ldots, u_l)$, $V = (v_1, v_2, \ldots, v_l) \in (\mathbb{C}^d)^l$ and $u_i, v_i \in \mathbb{C}^d (i = 1, 2, \ldots, l)$. Moreover, it is always assumed that each matrix norms, arising in the following, is subject to the corresponding vector norm.

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Consider the following systems of nonlinear NDDEs
\[
\begin{cases}
\frac{d}{dt}[y(t) - Ny(t - \tau)] = f(t, y(t), y(t - \tau)), & t \geq 0, \\
y(t) = \phi(t), & -\tau \leq t \leq 0,
\end{cases}
\]
(2.1)
and
\[
\begin{cases}
\frac{d}{dt}[z(t) - Nz(t - \tau)] = f(t, z(t), z(t - \tau)), & t \geq 0, \\
z(t) = \psi(t), & -\tau \leq t \leq 0,
\end{cases}
\]
(2.2)
where \(\tau > 0\) is constant delay, \(N \in \mathbb{C}^{d \times d}\) stands for a constant matrix with \(\|N\| < 1\), \(\phi, \psi: [-\tau, 0] \to \mathbb{C}^{d}\) are continuous functions, and \(f: [0, +\infty) \times \mathbb{C}^{d} \times \mathbb{C}^{d} \to \mathbb{C}^{d}\) is a given mapping subject to
\[
\text{Re}\langle (x_1 - x_2, -N(y_1 - y_2), f(t, x_1, y_1) - f(t, x_2, y_2)) \rangle \\
\leq \alpha\|x_1 - x_2\|^2 + \beta\|y_1 - y_2\|^2, & t \geq 0, \ x_1, x_2, y_1, y_2 \in \mathbb{C}^{d},
\]
(2.3)
in which \(\alpha, \beta\) are real constants.

The problems of the form (2.1) can be found in the systems with lossless transmission lines (cf. [16]). In the following all the problems (2.1) with (2.3) will be referred as the class \(R_{a,b}\). For instance, a complex d-dimensional linear system
\[
\begin{cases}
\frac{d}{dt}[y(t) - Ny(t - \tau)] = Ly(t) + My(t - \tau), & t \geq 0, \\
y(t) = \phi(t), & -\tau \leq t \leq 0,
\end{cases}
\]
belongs to the class \(R_{a,b}\) whenever the matrix
\[
G = \frac{1}{2} \begin{pmatrix} L + L^* - 2\alpha I & M - L^*N \\ M^* - N^*L & -N^*M - M^*N - 2\beta I \end{pmatrix}
\]
is negative definite, where \(I\) denote a d-dimensional identity matrix and * is the conjugate transpose symbol of the matrices, since
\[
\text{Re}\langle (x_1 - x_2, -N(y_1 - y_2), L(x_1 - x_2) + M(y_1 - y_2)) \rangle \\
-\alpha\|x_1 - x_2\|^2 - \beta\|y_1 - y_2\|^2 \\
= \langle (x_1 - x_2, y_1 - y_2), G(x_1 - x_2, y_1 - y_2) \rangle, & \forall x_1, x_2, y_1, y_2 \in \mathbb{C}^{d}.
\]

For the problems of the class \(R_{a,b}\), we obtain the following stability results.

**Theorem 2.1.** Suppose problems (2.1), (2.2) belong to the class \(R_{a,b}\) with
\[
\alpha \leq 0, \quad \beta \leq \alpha\|N\|^2.
\]
Then we have
(a) \(\|y(t) - z(t)\| \leq \frac{2}{\alpha\|N\|} \max_{-\tau \leq \theta \leq 0} \|\phi(\theta) - \psi(\theta)\|, \ t \geq 0,\)
(b) for \(\alpha < 0\), \(\lim_{t \to +\infty} \|y(t) - z(t) - N(y(t - \tau) - z(t - \tau))\| = 0.

**Proof.** Let
\[
u(t) = y(t) - z(t), \quad \nu(t) = \|u(t) - Nu(t - \tau)\|^2, \\
F(t) = f(t, y(t), y(t - \tau)) - f(t, z(t), z(t - \tau)).
\]
Then by (2.3)
\[
\nu'(t) = 2\text{Re}\langle u(t) - Nu(t - \tau), F(t) \rangle \\
\leq 2\alpha\|u(t)\|^2 + \beta\|u(t - \tau)\|^2, & t \geq 0,
\]