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# Regular Splitting and Potential Reduction Method for Solving Quadratic Programming Problem with Box Constraints*[1)]

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**Abstract**

A regular splitting and potential reduction method is presented for solving a quadratic programming problem with box constraints (QPB) in this paper. A general algorithm is designed to solve the QPB problem and generate a sequence of iterative points. We show that the number of iterations to generate an $\epsilon$-minimum solution or an $\epsilon$-KKT solution by the algorithm is bounded by $O(\frac{n^2}{\epsilon}log\frac{1}{\epsilon}+nlog(1+\sqrt{2n}))$, and the total running time is bounded by $O(n^2(n+logn+log\frac{1}{\epsilon})(\frac{n}{\epsilon}log\frac{1}{\epsilon}+logn))$ arithmetic operations.

*Key words*: Quadratic programming problem, Regular splitting, Potential reduction algorithm, Complexity analysis.

## 1. Introduction

In this paper, we consider a specail form of a quadratic programming problem with box constrained variables(QPB) as follows:

$$QPB: \qquad min\ q(x) \quad \text{s.t.}\ (x,s)\in\Omega$$

where $\Omega=\{(x,s)\in R^n\times R^n: x+s=e,\ x\geq 0,\ s\geq 0\}$ is the feasible region of the problem and $s$ is a slack vector, and $\Omega^0$ denotes the set of interior points of $\Omega$, and $q(x)=\frac{1}{2}x^THx+c^Tx$, and $H\in R^{n\times n}$ is a symmetric matrix, and $c,\ e\in R^n$ are given vectors and all the elements of $e$ are one. Without loss of generality, if the constrained variables of a quadratic programming problem with box constraints are bounded, then the problem can be transformed into the QPB specail form.

This problem arises in several areas of applications, such as problem of differential equations, discrete optimal control with continue time and design engineering, linear least square problem with box constraints or as a sequential subproblem of nonlinear programming methods. Therefore, it has a special importance.

Many different algorithms have been studied for solving this type of problem, such as projection gradient method[1], active-set method[12], matrix splitting methods[2,3,9], and the interior point method[10,11]. If the QPB problem is a convex problem, then it can be solved in polynomial time. If the QPB problem is a nonconvex problem, then it becomes a hard problem–NP complete problem. Some of algorithms can be also used to solve the problem, but it is difficult to obtain a global or local minimal solution[5-8]. On the other hand, searching a local minimum or checking the existence of a KKT point are an NP complete problem for a class of nonconvex optimization problems[7]. Therefore, $\epsilon$-approximate minimizer or $\epsilon$-KKT point was introduced in combinatorial optimization[6,7]. Finding an $\epsilon$-minimizer or $\epsilon$-KKT point is also

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hard problem. The complexity of finding an $\epsilon$-approximate minimizer or $\epsilon$-KKT point have been studied by many authors, and some of the results have been used in practice[11]. It would be mentioned that the steepest-descent-type method was used to compute an $\epsilon$-KKT point of the QPB problem, and the complexity of the algorithm was analyzed, and the arithmetic operations of the algorithm was bounded by $O(n^3(\frac{L}{\epsilon})^2)$, where $L$ is a fixed number depending on the problem data[6,7]. Other results are also discussed in [11].

In this paper, we present a regular splitting and potential reduction method for solving the QPB problem. The goal of the paper is to try finding a easy way to solve the problem. The main idea of the algorithm is to introduce a potential function for the original QPB problem and split the matrix $H$ into the sum of two matrices $H_1$ and $H_2$ such that $(H_1 - H_2)$ is a symmetric positive definite matrix, and a new minimization problem with Hessian matrix $H_1$ and an ellipsoid constraint is considered instead of solving the original QPB problem. The potential reduction techniques are used to solve the new problem such that the value of the potential function is reduced by a constant at each iteration. An $\epsilon$-minimum solution and $\epsilon$-KKT solution for QPB problem is defined, respectively. A general algorithm is designed to solve the QPB problem and generates a sequence of iterative points. We show that the number of total iterations to generate an $\epsilon$-minimum solution or an $\epsilon$-KKT solution by the algorithm is bounded by $O(\frac{n^2}{\epsilon}log\frac{1}{\epsilon} + nlog(1+\sqrt{2n}))$, and the total running time is bounded by $O(n^2(n + logn + log\frac{1}{\epsilon})(\frac{n}{\epsilon}log\frac{1}{\epsilon} + logn))$ arithmetic operations.

## 2. Regular splitting and potential reduction algorithm

The regular splitting and potential reduction algorithm for solving the QPB problem will be described in this section. For the sake of convenience, some of definitions and the basic results are firstly introduced.

**Proposition 1.** $(x^*, s^*) \in R^n \times R^n$ is a minimum solution of the QPB problem, then there is $(y, z) \in R^n \times R^n$ such that the following relationships hold

$$x^* + s^* = e, \quad x^* \geq 0, \quad s^* \geq 0, \tag{2.1a}$$

$$Hx^* + c + y - z = 0, \quad y \geq 0, \quad z \geq 0, \tag{2.1b}$$

$$y^T x^* = 0, \quad z^T s^* = 0. \tag{2.1c}$$

The formula (2.1) is the first order optimality conditions or KKT condition of the QPB problem. Let $\bar{\Omega} = \{(x,y,z) \in R^n \times R^n \times R^n : Hx + c + y - z = 0, x \geq 0, y \geq 0, z \geq 0\}$. Thus, $\bar{\Omega}$ is the set of dual feasible region of the QPB problem.

**Definition 1.** $(H_1, H_2) \in R^{n\times n} \times R^{n\times n}$ is said to be a regular splitting of $H \in R^{n\times n}$ if (i) $H = H_1 + H_2$ and (ii)$(H_1 - H_2)$ is a positive definite matrix.

Let $l_e$ and $u_e$ denote the minimal and maximal objective value of the QPB problem on $\Omega$, respectively. Then we can define an $\epsilon$-minimal solution or $\epsilon$-KKT solution of the QPB problem, respectively.

**Definition 2.** $(x, s) \in \Omega$ is said to be an $\epsilon$-minimum solution of the QPB problem, $\epsilon \in (0,1)$ if $\frac{q(x)-l_e}{u_e-l_e} \leq \epsilon$. Similarly, $(x,s) \in \Omega$ is said to be an $\epsilon$-KKT solution for the QPB problem if $(x,y,z) \in \bar{\Omega}$, and $\frac{x^Ty+s^Tz}{u_e-l_e} \leq \epsilon$.

As is well known, the potential reduction algorithm is usually required to start at an analytic center point or an approximate analytic center point of the feasible region for the solved problem. So, it is easy to show that $x^0 = \frac{1}{2}e$ and $s^0 = \frac{1}{2}e$ are the analytic center point of the feasible region $\Omega$, and that there are two ellipsoids $V_1$ and $V_2$ such that $\Omega \supset V_1 = \{(x,s) \in \Omega, ||(X^0)^{-1}(x - x^0)||^2 + ||(S^0)^{-1}(s - s^0)||^2 \leq 1\}$, and $\Omega \subset V_2 = \{(x,s) \in \Omega, ||(X^0)^{-1}(x - x^0)||^2 + ||(S^0)^{-1}(s - s^0)||^2 \leq 2n\}$. Where $X, S$ denote the diagonal matrices with elements of $x, s$, respectively. In other word, $\Omega$ is inscribed and outscribed by $V_1$ and $V_2$, respectively. Thus, we have the following conclusion.