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# HIGH-ORDER I-STABLE CENTERED DIFFERENCE SCHEMES FOR VISCOUS COMPRESSIBLE FLOWS*

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**Dedicated to the 80th birthday of Professor Zhou Yulin**

**Abstract**

In this paper we present high-order I-stable centered difference schemes for the numerical simulation of viscous compressible flows. Here I-stability refers to time discretizations whose linear stability regions contain part of the imaginary axis. This class of schemes has a numerical stability independent of the cell-Reynolds number $Rc$, thus allows one to simulate high Reynolds number flows with relatively larger $Rc$, or coarser grids for a fixed $Rc$. On the other hand, $Rc$ cannot be arbitrarily large if one tries to obtain adequate numerical resolution of the viscous behavior. We investigate the behavior of high-order I-stable schemes for Burgers' equation and the compressible Navier-Stokes equations. We demonstrate that, for the second order scheme, $Rc \leq 3$ is an appropriate constraint for numerical resolution of the viscous profile, while for the fourth-order schemes the constraint can be relaxed to $Rc \leq 6$. Our study indicates that the fourth order scheme is preferable: better accuracy, higher resolution, and larger cell-Reynolds numbers.

*Key words*: I-stable, Viscous compressible flow, Burgers' equation, Cell-Reynolds number constraint.

## 1. Introduction

Compressible flows with high Reynolds numbers, or, more generally, systems of conservation laws with small viscosities, remain a challenging numerical problem, even with great progress in the development of modern shock capturing methods for inviscid flows (the Euler equations) or systems of conservation laws in the last two decades. On the one hand, due to the constraint on the computing capacity, one attempts to simulate high Reynolds number flow with relative coarse grids (larger cell-Reynolds numbers), but on the other hand, when the cell-Reynolds number becomes too large, one loses appropriate resolution on the viscous effect and the numerical solutions become unphysical.

Due to the great success of modern shock capturing methods for hyperbolic systems, a very natural idea for the simulation of the viscous flows seems to be the application of a shock capturing method for the convection terms, coupled with some centered differences for the viscosity term. By building a numerical viscosity into the scheme which reduces the accuracy to first order across the discontinuities in order to suppress the numerical oscillations, shock

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capturing schemes are very effective in simulating inviscid flows and hyperbolic systems of conservation laws [13]. Since the main idea of shock-capturing is underresolution across the discontinuity, when simulating a slightly viscous flows where viscous effect is important, the mixture of numerical viscosities with the physical ones become a subtle issue.

In this paper, we seek an alternative approach by using simply high order centered difference schemes. This approach allows zero numerical viscosity, thus guarantees that, under enough resolution, the viscous effect observed numerically is purely physical. However, the traditional second order centered difference schemes for slightly viscous convection equations has a cell-Reynolds number constraint $Rc \leq 2$. To break this stability barrier, we take the method of line approach and use the so-called I-stable time discretization, which yields a numerical stability independent of $Rc$.

We call that a time-discretization for an ordinary differential equation is **I-stable** if the linear stability region contains part of the imaginary axis. In [19] Vichnevetsky studied the stability charts in the numerical approximation of partial differential equations. He first found that the linear stability regions of some time-discretization schemes contain part of the imaginary axis and applied these schemes to linear hyperbolic and advection-diffusion equations. In [3, 4], E and Liu realized that to solve the incompressible Navier-Stokes equations, using the fourth-order Runge-Kutta method (which is I-stable according to the above definition) along with a fourth-order centered difference for the convection removes the cell-Reynolds number constraint, thus allows them to simulate incompressible flows with very high Reynolds numbers. Choi and Liu [2] introduced a class of three-stage, second order Runge-Kutta method (which is I-stable) for the compressible Euler equations, and observed good convergence property toward the steady-state solution. This is the scheme we will explore here.

While the I-stable scheme has a remarkable stability property, which allows one to use an arbitrarily large cell-Reynolds number, in practice, this can never be done if one wants to resolve the viscous effect. Failing to resolve adequately the viscous effect will simply produce the results for the inviscid equations, rather than the viscous equations. The rule of the game is to use relatively larger cell-Reynolds number (if stability allows) but still resolve the viscous effect without numerical oscillations. It is the goal of this paper to investigate the suitable cell-Reynolds number constraint for viscous conservation laws using high-order I-stable centered differences. We use the Burgers' equation and the compressible Navier-Stokes equations as examples to study this issue.

We observe that, when using a second-order I-stable centered difference scheme, $Rc \leq 3$ is an appropriate constraint, while for the forth-order I-stable schemes this can be relaxed to $Rc \leq 6$. Within this range of $Rc$ the numerical schemes are stable and are essentially non-oscillatory. This significantly improves the traditional cell-Reynolds number constraint $Rc \leq 2$, and sheds light on a promising direction to develop numerical schemes for compressible flows with high Reynolds numbers.

This paper is organized as follows: In the next section, we introduce the high-order I-stable centered difference for viscous conservation laws. In section 3, using the Burgers' equation as an example, we investigate the effect of the cell-Reynolds number for different time and spatial discretizations. In section 4, we propose the fourth-order I-stable scheme for the 2-D compressible Navier-Stokes equations. We study numerically the effect of cell-Reynolds number using flows in a driven cavity and a Buoying-driven cavity. We end in section 5 with some discussions.

## 2. For systems of conservation laws with small viscosity

Consider the scalar conservation laws with viscosity:

$$\partial_t u + \partial_x f(u) = \nu \partial_{xx} u \, . \tag{2.1}$$