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NONCONFORMING QUADRILATERAL ROTATED Q_1 ELEMENT FOR REISSNER-MINDLIN PLATE *1)

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Dedicated to the 80th birthday of Professor Zhou Yulin

Abstract

In this paper, we extend two rectangular elements for Reissner-Mindlin plate [9] to the quadrilateral case. Optimal H^1 and L^2 error bounds independent of the plate hickness are derived under a mild assumption on the mesh partition.

Key words: Reissner-Mindlin Plate, Quadrilateral Rotated Q_1 element, Locking-free.

1. Introduction

We consider the finite element approximation of the solution of Reissner-Mindlin (R-M hereinafter) model, which describes the deformation of a plate subjected to a transverse loading in terms of the transverse displacement of the midplane and the rotations of fibers normal to the midplane. As it is well-known, standard finite element approximation of this model usually fails to yield good results when the plate thickness is small, which is commonly referred to *locking phenomenon*, so some numerical stabilization tricks such as reduced integration or the mixed variational principles are needed to overcome this difficulty. MS elements proposed in [9] seem the simplest rectangular elements in such category [3]. However, quadrilateral elements are far more flexible than rectangular elements, so it is quite important to construct quadrilateral R-M plate elements, or extend the existing rectangular R-M elements to the quadrilateral case. On the other and, it is noticed recently that the extension of rectangular R-M elements to isoparametric quadrilateral R-M elements is not so straightforward [10]. The goal of this paper is to extend MS elements to the quadrilateral case and give a mathematical analysis.

We conclude this section with a list of some basic notations used in the sequel. In §2, the R-M plate model and its varational formulation of Brezzi and Fortin [4, 6] are recalled. In §3, we describe the quadrilateral version of MS elements and the method we used is recast in the variational formulation of Brezzi and Fortin based upon a kind of discrete Helmholtz Decomposition. The error estimates are included in §4.

We use the standard notation and definition for the Sobolev spaces $H^s(\Omega)$ and $H^s(\partial\Omega)$ for $s \geq 0$ [1], the standard associated inner products are denoted by $(\cdot,\cdot)_s$ and $(\cdot,\cdot)_{s,\partial\Omega}$, and their norms by $\|\cdot\|_s$ and $\|\cdot\|_{s,\partial\Omega}$, respectively. For $s=0$, $H^s(\Omega)$ coincides with a $L^2(\Omega)$. In this case, the norm and inner product are denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)$ respectively. As usual, $H^s_0(\Omega)$ is the subspace of $H^s(\Omega)$ with vanishing trace on Ω. Let $L^2_0(\Omega)$ be the set of all $L^2(\Omega)$ functions with zero integral mean.

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Throughout this paper, the generic constant C is assumed to be independent of the plate thickness t and the mesh size h.

Finally, we use the standard differential operators:

$$\nabla r = \begin{pmatrix} \partial r/\partial x \\ \partial r/\partial y \end{pmatrix}, \qquad \operatorname{curl} p = \begin{pmatrix} \partial p/\partial y \\ -\partial p/\partial x \end{pmatrix},$$

$$\operatorname{div} \boldsymbol{\psi} = \partial\psi_1/\partial x + \partial\psi_2/\partial y, \quad \operatorname{rot} \boldsymbol{\psi} = \partial\psi_2/\partial x - \partial\psi_1/\partial y.$$

We also need the following vector spaces

$$\boldsymbol{H}_0(\operatorname{rot},\Omega) = \{\, \boldsymbol{q} \in \boldsymbol{L}^2(\Omega) \mid \operatorname{rot} \boldsymbol{q} \in L^2(\Omega), \boldsymbol{q}\cdot\boldsymbol{t} = 0 \text{ on } \partial\Omega \,\},$$

where $\boldsymbol{t}$ is denoted as the unit tangent to $\partial\Omega$, and

$$\boldsymbol{H}(\operatorname{div},\Omega) = \{\, \boldsymbol{q} \in \boldsymbol{L}^2(\Omega) \mid \operatorname{div} \boldsymbol{q} \in L^2(\Omega) \,\}.$$

The norm in $\boldsymbol{H}(\operatorname{div},\Omega)$ is given by

$$\|\boldsymbol{\eta}\|_{\boldsymbol{H}(\operatorname{div})} = \left(\|\boldsymbol{\eta}\|_0^2 + \|\operatorname{div}\boldsymbol{\eta}\|_0\right)^{1/2}.$$

2. Reissner-Mindlin Plate Model

Let Ω be a convex polygon representing the mid-surface of the plate. Assume that the plate is clamped along the boundary $\partial\Omega$. Let ω and $\boldsymbol{\phi}$ denote the transverse deflection and the rotations, respectively, which are determined by the following

Problem 2.1. *Find* $(\boldsymbol{\phi},\omega) \in \boldsymbol{H}_0^1(\Omega)\times H_0^1(\Omega)$ *such that*

$$a(\boldsymbol{\phi},\boldsymbol{\psi}) + (\boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi}) = (g, v) \quad \forall(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0^1(\Omega)\times H_0^1(\Omega). \tag{2.1}$$

The shear strain $\boldsymbol{\gamma}$ *is defined as*

$$\boldsymbol{\gamma} := \lambda t^{-2}(\nabla\omega - \boldsymbol{\phi}).$$

Here g is the scaled transverse loading, t is the plate thickness, $\lambda = E\kappa/2(1+\nu)$ is the shear modulus with Young's modulus E, ν the Poisson ratio, and κ the shear correction factor. The bilinear form a is defined as $a(\boldsymbol{\eta},\boldsymbol{\psi}) = (\mathcal{C}\mathcal{E}\boldsymbol{\eta}, \mathcal{E}\boldsymbol{\psi})$, here $\mathcal{C}\tau$ is defined for any 2×2 symmetric matrix τ as

$$\mathcal{C}\tau := \frac{E}{12(1-\nu^2)}\left[(1-\nu)\tau + \nu \operatorname{tr}(\tau)\boldsymbol{I}\right].$$

Following [4] and [6], Problem 2.1 can be written into the following decoupled system as

Problem 2.2. *Find* $(r,\boldsymbol{\phi},p,\boldsymbol{\alpha},\omega) \in H_0^1(\Omega)\times\boldsymbol{H}_0^1(\Omega)\times L_0^2(\Omega)\times\boldsymbol{H}_0(\operatorname{rot},\Omega)\times H_0^1(\Omega)$*, such that*

$$\begin{aligned}
(\nabla r, \nabla\mu) &= (g,\mu) \quad \forall\mu \in H_0^1(\Omega),\\
a(\boldsymbol{\phi},\boldsymbol{\psi}) - (p, \operatorname{rot}\boldsymbol{\psi}) &= (\nabla r, \boldsymbol{\psi}) \quad \forall\boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega),\\
-(\operatorname{rot}\boldsymbol{\phi}, q) - \lambda^{-1}t^2(\operatorname{rot}\boldsymbol{\alpha}, q) &= 0 \quad \forall q \in L_0^2(\Omega),\\
(\boldsymbol{\alpha},\boldsymbol{\delta}) - (p, \operatorname{rot}\boldsymbol{\delta}) &= 0 \quad \forall\boldsymbol{\delta} \in \boldsymbol{H}_0(\operatorname{rot},\Omega),\\
(\nabla\omega, \nabla s) &= (\boldsymbol{\phi} + \lambda^{-1}t^2\nabla r, \nabla s) \quad \forall s \in H_0^1(\Omega).
\end{aligned}$$