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ARTIFICIAL BOUNDARY CONDITIONS FOR "VORTEX IN CELL" METHOD*1)

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Dedicated to the 80th birthday of Professor Zhou Yulin

Abstract

This paper mainly designs artificial boundary conditions for "vortex in cell" method in solving two-dimensional incompressible inviscid fluid under two conditions: one is with periodical initial value in one direction and the other with compact supported initial value. To mimic the vortex motion, Euler equation is transformed into vorticity-stream function and the technique of vortex in cell is applied incorporating with the artificial boundary conditions.

Key Words: Incompressible inviscid flow, Vortex in cell method, Artificial boundary condition

1. Mathematical Model

The motion of two-dimensional incompressible inviscid fluid satisfies the following equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho}\nabla P = \vec{f} \tag{1.1}$$

$$\nabla \boldsymbol{u} = 0, \tag{1.2}$$

where $\boldsymbol{u}(u,v)$ is the velocity vector of a particle at $\vec{x}=(x,y)$ and time t. $P(\vec{x},t)$ and $\rho(\vec{x},t)$ denote the pressure and the density of the fluid respectively. $\rho(x,t)$ is constant for incompressible fluid. Fone $\vec{f}$ acts on a unit fluid. ∇ is gradient operator. '$\cdot$' stands for the inner product. In (x,y)-plane $\nabla\Lambda = (\frac{\partial}{\partial y}, -\frac{\partial}{\partial x})$ is rotation operator. Here we assume $\vec{f}=\nabla\varphi$, where φ is a scalar function, and define vorticity ω as $\omega = -\nabla\Lambda\boldsymbol{u} = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$. Inserting $\nabla\Lambda$ into Euler equation (1.1), one can obtain:

$$\frac{\partial \omega}{\partial t} + (\boldsymbol{u}\cdot\nabla)\omega = 0. \tag{1.3}$$

Let $\vec{x}_0$ be an arbitrary point in the plane and define the stream function $\psi(\vec{x})$ as

$$\psi(\vec{x}) = \int_{\vec{x}_0}^{\vec{x}} v dx + u dy,$$

where the integration in (x,y) plane is independent of the integration path due to (**??**) and Green formula. Applying $\nabla\Lambda$ and $-\nabla\Lambda$ to $\psi(\vec{x})$ in succession, one can get the relationship of

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the stream function and vorticity, i. e., $-\Delta\psi = \omega$. Thus the incompressible inviscid fluid in vorticity-stream function is:

$$\frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y} = 0 \tag{1.4}$$

$$-\Delta\psi = \omega \tag{1.5}$$

$$\frac{\partial\psi}{\partial y} = u \tag{1.6}$$

$$\frac{\partial\psi}{\partial x} = -v. \tag{1.7}$$

2. Technique of "vortex in cell"

Christansen [?] is the first one to use the technique of vortex in cell to simulate the motion of two-dimensional incompressible inviscid fluid. This paper adopts this method to simulate vortices interaction in (x, y)-plane. We discrete the fluid vorticity ω into sum of vortex points:

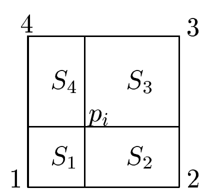
$$\omega(x, y) \approx \sum_{i=1}^{M} p_i \delta(x - x_i)\delta(y - y_i),$$

where M is the total number of vortex points, p_i is vorticity of the i-th point at (x_i, y_i). The velocity $\boldsymbol{u}_i = (u_i, v_i)$ of the i-th vortex point is given by

$$\frac{\partial x_i}{\partial t} = u_i \tag{2.1}$$

$$\frac{\partial y_i}{\partial t} = v_i. \tag{2.2}$$

Once we know the velocity (u_i, v_i) of the vortex point, then we can advance the vortex point by using (**??**)-(**??**). Vortex in cell method solves the Poisson's equation (**??**) on a uniform mesh by the usual five-point scheme. Thus to mimic the motion of vortex points, it is necessary to allocate the vortex point's vorticity to mesh points first and then, after solving the Poisson's equation (**??**) on uniform mesh, to redistribute mesh point's velocity to vortex points. Let s be the area of an mesh and s_1, s_2, s_3, s_4 be the four parts divided by a vortex point located in the mesh showed by the following figure. Then the vortex point allocates its vorticity p_i to the four surrounding mesh points as follows:



$$\omega_1 = \frac{s_3}{s}p_i \quad \omega_2 = \frac{s_4}{s}p_i \quad \omega_3 = \frac{s_1}{s}p_i \quad \omega_4 = \frac{s_2}{s}p_i.$$

All other vortex points located in the same mesh allocate their vorticity to the mesh points similarly. On the other hand, mesh points distribute their velocity to the vortex point p_i by the technique of vortex in cell. That is, assume $\boldsymbol{v}_l,\ l = 1, 2, 3, 4$ are the velocities of the four surrounding points of a mesh , then the velocity $\boldsymbol{u}_i$ of the vortex point p_i located in the mesh is defined by

$$\boldsymbol{u}_i = \frac{s_3}{s}\boldsymbol{v}_1 + \frac{s_4}{s}\boldsymbol{v}_2 + \frac{s_1}{s}\boldsymbol{v}_3 + \frac{s_2}{s}\boldsymbol{v}_4.$$