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THE UNCONDITIONAL STABLE DIFFERENCE METHODS WITH INTRINSIC PARALLELISM FOR TWO DIMENSIONAL SEMILINEAR PARABOLIC SYSTEMS*1)

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Dedicated to the 80th birthday of Professor Zhou Yulin

Abstract

In this paper we are going to discuss the difference schemes with intrinsic parallelism for the boundary value problem of the two dimensional semilinear parabolic systems. The unconditional stability of the general finite difference schemes with intrinsic parallelism is justified in the sense of the continuous dependence of the discrete vector solution of the difference schemes on the discrete data of the original problems in the discrete $W_2^{(2,1)}$ norms. Then the uniqueness of the discrete vector solution of this difference scheme follows as the consequence of the stability.

Key words: Difference Scheme, Intrinsic Parallelism, Two Dimensional Semilinear Parabolic System, Stability.

1. Introduction

1. In this paper we consider the boundary value problems for the two dimensional semilinear parabolic systems of second order of the form

$$u_t = A(x,y,t)(u_{xx}+u_{yy}) + B(x,y,t,u)u_x + C(x,y,t,u)u_y + f(x,y,t,u) \tag{1}$$

where $(x,y) \in \Omega = (0,l_1)\times(0,l_2)$, $t\in(0,T]$, and $u(x,y,t) = (u_1(x,y,t), u_2(x,y,t), \cdots, u_m(x, y,t))$ is a m–dimensional vector unknown function ($m \geq 1$); $A(x,y,t)$, $B(x,y,t,u)$ and C (x, y, t, u) are given $m\times m$ matrix functions, and $f(x,y,t,u)$ is a m–dimensional vector function and $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $u_{yy} = \frac{\partial^2 u}{\partial y^2}$ and $u_t = \frac{\partial u}{\partial t}$ are the corresponding m–dimensional vector derivatives of the m–dimensional unknown vector function $u(x,y,t)$.

In the domain $Q_T = \bar{\Omega}\times[0,T]$ the homogeneous boundary conditions and the initial condition for the system (1) are

$$u(x,y,t) = 0, \quad (x,y)\in\partial\Omega, 0 < t \leq T, \tag{2}$$

$$u(x,y,0) = \varphi(x,y), \quad (x,y)\in\Omega. \tag{3}$$

In [1]–[9] the general finite difference schemes with intrinsic parallelism for the linear and quasilinear parabolic problems have been discussed. For the one-dimensional quasilinear parabolic systems, in [8] some general difference schemes with intrinsic parallelism are constructed and proved to be unconditional stable and convergent. For the two dimensional quasilinear parabolic systems, in [9] some fundamental behaviors of general finite difference schemes with intrinsic

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parallelism are studied, i.e., the existence of the discrete vector solutions of the nonlinear difference system with intrinsic parallelism is proved, and the convergence of the discrete vector solutions of the certain difference schemes with intrinsic parallelism to the unique generalized solution of the original quasilinear parabolic problem is proved.

2. Difference Schemes with Intrinsic Parallelism

2. Divide the domain $Q_T = \{0 \le x \le l_1, 0 \le y \le l_2, 0 \le t \le T\}$ into small grids by the parallel planes $x = x_i$ $(i = 0, 1, \cdots, I)$, $y = y_j$ $(j = 0, 1, \cdots, J)$ and $t = t^n$ $(n = 0, 1, \cdots, N)$ with $x_i = ih_1$, $y_j = jh_2$ and $t^n = n\tau$, where $Ih_1 = l_1$, $Jh_2 = l_2$ and $N\tau = T$, I, J and N are integers and h_1, h_2 and τ are the steplengths of grids. Denote $h^* = \max(h_1, h_2) = h$, $h_* = \min(h_1, h_2)$. Denote $v_\Delta = \{v_{ij}^n | i = 0, 1, \cdots, I; j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ the m–dimensional discrete vector function defined on the discrete rectangular domain $Q_\Delta = \{(x_i, y_j, t^n) | i = 0, 1, \cdots, I; j = 0, 1, \cdots, J; n = 0, 1, \cdots, N\}$ of the grid points.

Let us now construct the general difference schemes with intrinsic parallelism for the boundary value problem (1), (2) and (3):

$$\frac{v_{ij}^{n+1} - v_{ij}^n}{\tau} = A_{ij}^{n+1} \overset{*}{\Delta} v_{ij}^{n+1} + B_{ij}^{n+1} \bar{\delta}_x^1 v_{ij}^{n+1} + C_{ij}^{n+1} \bar{\delta}_y^1 v_{ij}^{n+1} + f_{ij}^{n+1}, \tag{$1)_\Delta$}$$
$$(i = 1, 2, \cdots, I-1; j = 1, 2, \cdots, J-1; n = 0, 1, \cdots, N-1),$$

where

$$\begin{aligned}
\overset{*}{\Delta} v_{ij}^{n+1} &= \overset{*}{\delta}{}_x^2 v_{ij}^{n+1} + \overset{*}{\delta}{}_y^2 v_{ij}^{n+1} \\
&= \frac{v_{i+1,j}^{n+\lambda_{ij}^n} - 2v_{ij}^{n+1} + v_{i-1,j}^{n+\mu_{ij}^n}}{h_1^2} + \frac{v_{i,j+1}^{n+\bar{\lambda}_{ij}^n} - 2v_{ij}^{n+1} + v_{i,j-1}^{n+\bar{\mu}_{ij}^n}}{h_2^2},
\end{aligned}$$

$$\begin{aligned}
A_{ij}^{n+1} &= A(x_i, y_j, t^{n+1}), \\
B_{ij}^{n+1} &= B(x_i, y_j, t^{n+1}, \tilde{\delta}^0 v_{ij}^{n+1}), \\
C_{ij}^{n+1} &= C(x_i, y_j, t^{n+1}, \hat{\delta}^0 v_{ij}^{n+1}), \\
f_{ij}^{n+1} &= f(x_i, y_j, t^{n+1}, \bar{\delta}^0 v_{ij}^{n+1}).
\end{aligned} \tag{4}$$

In this difference scheme, the expressions $\tilde{\delta}^0 v_{ij}^{n+1}$, $\hat{\delta}^0 v_{ij}^{n+1}$, $\bar{\delta}^0 v_{ij}^{n+1}$, and $\bar{\delta}_x^1 v_{ij}^{n+1}, \bar{\delta}_y^1 v_{ij}^{n+1}$ can be taken in the following manner. We can take

$$\begin{aligned}
\tilde{\delta}^0 v_{ij}^{n+1} = \; & \lambda_{ij}^n \alpha_{1ij}^n v_{i+1j}^{n+1} + \mu_{ij}^n \alpha_{2ij}^n v_{i-1j}^{n+1} + \bar{\lambda}_{ij}^n \alpha_{3ij}^n v_{ij+1}^{n+1} \\
& + \bar{\mu}_{ij}^n \alpha_{4ij}^n v_{ij-1}^{n+1} + \alpha_{5ij}^n v_{ij}^{n+1} + \bar{\alpha}_{1ij}^n v_{i+1j}^n \\
& + \bar{\alpha}_{2ij}^n v_{i-1j}^n + \bar{\alpha}_{3ij}^n v_{ij+1}^n + \bar{\alpha}_{4ij}^n v_{ij-1}^n + \bar{\alpha}_{5ij}^n v_{ij}^n
\end{aligned} \tag{5}$$

such that the sum of coefficients equals to unit, that is

$$\lambda_{ij}^n \alpha_{1ij}^n + \mu_{ij}^n \alpha_{2ij}^n + \bar{\lambda}_{ij}^n \alpha_{3ij}^n + \bar{\mu}_{ij}^n \alpha_{4ij}^n + \alpha_{5ij}^n + \bar{\alpha}_{1ij}^n + \bar{\alpha}_{2ij}^n + \bar{\alpha}_{3ij}^n + \bar{\alpha}_{4ij}^n + \bar{\alpha}_{5ij}^n = 1$$

and the sum of the absolute value of these coefficients is uniformly bounded by any given constant with respect to the indices i, j and n. The coefficients are dependent on the indices i, j and n, this means they are different for different layers and different grid points. This shows that the choice of the coefficients has great degree of freedom.

For the expressions $\bar{\delta}_x^1 v_{ij}^{n+1}$ and $\bar{\delta}_y^1 v_{ij}^{n+1}$, we can take for example as

$$\begin{aligned}
\bar{\delta}_x^1 v_{ij}^{n+1} = \; & d_{1ij}^n \frac{v_{i+1j}^{n+\lambda_{ij}^n} - v_{ij}^{n+1}}{h_1} + d_{2ij}^n \frac{v_{ij}^{n+1} - v_{i-1j}^{n+\mu_{ij}^n}}{h_2} \\
& + d_{3ij}^n \delta_x v_{ij}^n + d_{4ij}^n \delta_x v_{i-1j}^n,
\end{aligned}$$