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UNIFORM CONVERGENCE OF HERMITE INTERPOLATION OF HIGHER ORDER*[1)]

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Abstract

In this paper the uniform convergence of Hermite-Fejér interpolation and Grünwald type theorem of higher order on an arbitrary system of nodes are presented.

Key words: Hermite-Fejér interpolation, convergence, Hermit interpolation.

1. Introduction

Let us assume n, $n \geq 2$, m_{kn}, $k = 1, 2, 3, ..., n$, be integers and triangular matrix $X = \{x_{1n}, x_{2n}, ..., x_{nn}\}$, where

$$1 = x_{0n} \geq x_{1n} > x_{2n} > ... > x_{nn} \geq x_{n+1,n} = -1.$$

Let $\mathcal{N}_n = \sum_{k=1}^{n} m_{kn} - 1$, $m = \max_{n\geq 2, 1\leq k\leq n} m_{kn} < +\infty$, $\mathbf{N}_1=\{1, 3, 5, ...\}$, $\mathbf{N}_2=\{2, 4, 6, ...\}$ and $\mathbf{N}_0=\mathbf{N}_2 \bigcup\{0\}$. For simplicity we denote $\mathcal{N}_n$ as $\mathcal{N}$. In the following discussion we replace $x_{kn}, m_{kn}, k = 1, 2, ..., n$ with $x_k, m_k, k = 1, 2, ..., n$. Denoted by $\boldsymbol{P}_{\mathcal{N}}$ the set of polynomials of degree at most $\mathcal{N}$ and by A_{jk} the fundamental polynomials for Hermite interpolation of higher order, then we have $A_{jk} \in \boldsymbol{P}_{\mathcal{N}}$ satisfy

$$A_{jk}^{(p)}(x_q) = \delta_{jp}\delta_{kq}, \quad p = 0, 1, ..., m_q - 1, \quad j = 0, 1, ..., m_k - 1, \quad q, k = 1, 2, ..., n. \tag{1.1}$$

For $f \in C^r[-1, 1]$, $0 \leq r \leq m - 1$, the unique truncated Hermite interpolatory polynomial is given by

$$H_{nmr}(f, x) = \sum_{i=0}^{r}\sum_{k=1}^{n} f^{(i)}(x_k)A_{ik}(x), \tag{1.2}$$

here let $A_{jk} = 0$ if $j \geq m_k$. In particular, when $r = 0$ and $r = m - 1$ H_{nm0} and $H_{nm,m-1}$ are denoted by H_{nm} and H_{nm}^* respectively. We recognize that H_{n1} is the classical Lagrange interpolation and H_{n2} the classical Hermite-Fejér interpolation. H_{nmr} is called Lagrange type interpolation for odd m_k, $k = 1, 2, ..., n$, and Hermite-Fejér type interpolation for even m_k, $k = 1, 2, ..., n$, respectively.

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For giving the explicit expression of $A_{jk}(x)$ we let

$$L_k(x) = \prod_{q=1, q\neq k}^{n} \left(\frac{x - x_q}{x_k - x_q}\right)^{m_q}, \quad k = 1, 2, ..., n,$$

$$b_{vk} = \frac{1}{v!}\left[\frac{1}{L_k(x)}\right]^{(v)}_{x=x_k}, \quad v = 0, 1, ..., m_k - 1, \quad k = 1, 2, ..., n,$$

$$B_{jk}(x) = \sum_{v=0}^{m_k-j-1} b_{vk}(x - x_k)^v, \quad j = 0, 1, ..., m_k - 1, \quad k = 1, 2, ..., n.$$

Then by [6] it has

$$A_{jk}(x) = \frac{1}{j!}(x - x_k)^j B_{jk}(x) L_k(x), \quad 0 \leq j \leq m_k, \quad 1 \leq k \leq n. \tag{1.3}$$

More let

$$d_k = \max\{|x_k - x_{k+1}|, |x_k - x_{k-1}|\}, \quad k = 1, 2, ..., n,$$

$$D_n = \max_{1\leq k\leq n} d_k, \quad \|P\|_j := \max_{1\leq l\leq j} \|P^{(l)}\|,$$

$$R_{nm}(f, x) := |H_{nm}(f, x) - f(x)|,$$

$$r_{nm}(x) := R_{nm}(f_1, x) + R_{nm}(f_2, x), \quad f_i = x^i, i = 0, 1, 2, ...$$

$$S_{nm}(x) := \sum_{k=1}^{n} |(x - x_k)A_{0k}(x)|.$$

In what follows we denote by c, c_1, ..., positive constants independent of variables and indices, unless otherwise indicated; their value may be different occurrences even in subsequent formulas.

2. Main Result

In [4], Y. G. Shi has proved an important theorem about fundamental polynomials A_{jk}, B_{jk} as following:

Theorem A ([4, Theorem 2.1]). *If for a fixed n, $m_k - j$ is odd and $j < i \leq m_k - 1$ then*

$$B_{jk}(x) \geq c\left|\frac{x - x_k}{d_k}\right|^{i-j} |B_{ik}(x)|, \quad x \in \Re, \quad 1 \leq k \leq n, \tag{2.1}$$

and

$$|A_{ik}(x)| \leq c_1 d_k^{i-j} |A_{jk}(x)|, \quad x \in \Re, \quad 1 \leq k \leq n, \tag{2.2}$$

hold, where c and c_1 are positive constants depending only on m.

More the estimate of $R_{nm}(P, x)$ for all polynomials for the case of $m_k \equiv m$, $k = 1, 2, ..., n$ is given, that is the following theorem:

Theorem B ([4, Theorem 4.1]). *Let $m_k \equiv m$ be an even integer. Then for any $P \in P_{\mathcal{N}}$*

$$R_{nm}(P, x) \leq c\|P\|_m \left\{r_{nm}(x) + \frac{\|r_{nm}\| \ln^{10}[n(1 + \|r_{nm}\|)]}{n}\right\}, \tag{2.3}$$

where c depends only on m. Further more, if

$$\|H_{nm}\| = \left\|\sum_{k=1}^{n} |A_{0k}|\right\| = O(1) \tag{2.4}$$