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LEAST-SQUARES SOLUTIONS OF $X^TAX = B$ OVER POSITIVE SEMIDEFINITE MATRIXES A *1)

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Abstract

This paper is mainly concerned with solving the following two problems:
Problem I. Given $X \in R^{n\times m}$, $B \in R^{m\times m}$. Find $A \in P_n$ such that

$$\|X^TAX - B\|_F = \min,$$

where $$P_n = \{A \in R^{n\times n} | \quad x^TAx \geq 0, \quad \forall x \in R^n\}.$$

Problem II. Given $\tilde{A} \in R^{n\times n}$. Find $\hat{A} \in S_E$ such that

$$\|\tilde{A} - \hat{A}\|_F = \min_{A\in S_E} \|\tilde{A} - A\|_F,$$

where $\|\cdot\|_F$ is Frobenius norm, and S_E denotes the solution set of Problem I.

The general solution of Problem I has been given. It is proved that there exists a unique solution for Problem II. The expression of this solution for corresponding Problem II for some special case will be derived.

Key words: positive semidefinite matrix, Least-square problem, Frobenins norm

1. Introduction

[2] pointed out that $X^TAX = B$ comes from an inverse problem vibration theory. [2] has studied least-squares solutions where the unknown **A** is symmetric positive semidefinite, given the expression of general solution. It is more difficult to study least-squares solutions for the case that the unknown **A** is positive semidefinite (may be unsymmetric). In this paper we will discuss this problem. We will give the expression of general solution. Then we will discuss so called optimal approximation problem associated with $X^TAX = B$. That is: to find the optimal approximate of a given matrix $\tilde{A}$ by $A \in S_E$, where S_E is the solution set of the least-square problem of $X^TAX = B$. The existence and uniqueness of the solution for the problem is proved, the expression of the solution is derived for some conditions.

We denote the real $n \times m$ matrix space by $R^{n\times m}$, and $R^n = R^{n\times 1}$, the set of all matrices in $R^{n\times m}$ with rank r by $R_r^{n\times m}$, the set of all $n \times n$ orthogonal matrices by $OR^{n\times n}$, the set of all $n \times n$ symmetric matrices by $SR^{n\times n}$, the set of all $n \times n$ anti-symmetric matrices by $ASR^{n\times n}$, the column space, the null space and the Moore–penrose generalized inverse of a matrix A by $R(A), N(A), A^+$ respectively, the identity matrix of order n by I_n, the Frobenius norm of A by $\|A\|_F$. We define inner product in space $R^{n\times m}$, $(A, B) = \mathrm{tr}B^TA = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ij}, \quad \forall A, B \in R^{n\times m}$. Then $R^{n\times m}$ is a Hilbert inner product space. The norm of a matrix defined by the inner product is Frobenius norm.

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Definition 1. $A \in R^{n\times n}$ is called positive semidefinite if $x^TAx \geq 0$ for every non-vanishing vector x in R^n and denoted by $A \geq 0$.

Let

$$P_n = \{A \in R^{n\times n} | x^TAx \geq 0, \quad \forall x \in R^n\},$$
$$SR_{\geq}^{n\times n} = \{A \in R^{n\times n} | \quad A = A^T, \quad x^TAx \geq 0, \quad \forall x \in R^n\}.$$

Now we consider the following problems:

Problem I. Given $B \in R^{m\times m}, X \in R_r^{n\times m}$. Find $A \in P_n$ such that

$$f(A) = \|X^TAX - B\|_F = \min.$$

Problem II. Given $\tilde{A} \in R^{n\times n}$. Find $\hat{A} \in S_E$ such that

$$\|\tilde{A} - \hat{A}\|_F = \min_{A\in S_E} \|\tilde{A} - A\|_F,$$

where S_E is the solution set of Problem I.

In [2] the symmetric positive semidefinite and positive definite real solutions of $\|X^TAX - B\|_F = \min$ have been considered. And Problem II has not been studied (or the optimal approximate solution has not been studied).

At first, in this paper, we will discuss the optimal approximate problem on P_n. Then we will give the general solution of Problem I. At last, we will prove that there exists a unique solution for Problem II and derive the expression of this solution for some special case.

2. THE OPTIMAL APPROXIMATE PROBLEM ON P_n

Problem MA. Given nonempty closed convex cone $S \subseteq R^{n\times n}$, $F \in R^{n\times n}$, $D = \text{diag}\,(d_1, \cdots, d_n)$, $d_i > 0$, $i = 1, \cdots, n$. Find $\hat{E} \in S$ such that

$$\|D(\hat{E} - F)D\| \leq \|D(E - F)D\|, \quad \forall E \in S.$$

To solve Problem MA we introduce a conclusion.

Lemma 2.1[4]**.** *Suppose V is a real Hilbert space, $(\cdot,\cdot)$ denotes inner product , $\|u\|_V = \sqrt{(u,u)}$ represents norm in V, $K \subset V$ is a nonempty closed convex cone. $K^\perp$ represents the set of all elements which are orthogonal to K in V. It is obvious that $K^\perp$, $K^{\perp\perp} \triangleq (K^\perp)^\perp$ are closed linear subspace in V. $K^{\perp\perp}$ is the minimum subspace that concludes K. K^* is the dual cone of K in $K^{\perp\perp}$. Then, for every $u \in V$, there is an unique $u_0 \in K^\perp, u_1 \in K, u_2 \in K^*$ such that*

$$(u_1, u_2) = 0, \qquad u = u_0 + u_1 - u_2$$

and

$$\|u - u_1\|_V \leq \|u - v\|_V, \quad \forall v \in K$$

In $R^{n\times n}$ we define a new inner product and norm:

$$(A,B)_D = (DAD, DBD) = tr(DB^TD^2AD), \quad \|A\|_D = \sqrt{(A,A)_D} = \sqrt{(DAD,DAD)}$$

where $D = \text{diag}(d_1, \cdots, d_n)$, $d_i > 0$, $i = 1, \cdots, n$. This new Euclidean space is noted by $R_D^{n\times n}$. Therefore Problem MA is equivalent to

$$\|\hat{E} - F\|_D \leq \|E - F\|_D, \quad \forall E \in S \subseteq R_D^{n\times n}.$$

We point out P_n as a closed convex cone with vertex at zero point. In fact, it is evident that P_n is closed. And for any $\alpha \geq 0, \beta \geq 0$, there is $\alpha P_n + \beta P_n \subseteq P_n$. According to the definition of convex cone P_n is a closed convex cone.

By Lemma 2.1 Problem MA has a unique optimal approximate solution.

Lemma 2.2[4]**.** *For every matrix F of order n there are an anti-symmetric matrix F_0, a symmetric nonnegative definite matrix F_+ and a symmetric nonpositive definite matrix F_- such that*

$$F = F_0 + F_+ + F_-,$$