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FOURIER-CHEBYSHEV COEFFICIENTS AND GAUSS-TURÁN QUADRATURE WITH CHEBYSHEV WEIGHT[*1)]

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Abstract

The main purpose of this paper is to derive an explicit expression for Fourier-Chebyshev coefficient $A_{kn}(f) = \frac{2}{\pi}\int_{-1}^{1} f(x)T_{kn}(x)\frac{dx}{\sqrt{1-x^2}}, k, n \in \mathcal{N}_0$, which is initiated by L.Gori and C.A.Micchelli.

Key words: Fourier-Chebyshev coefficient, Gauss-Turán quadrature

1. Introduction

Throughout this paper let $x_1, \cdots, x_n$ be zeros of the Chebyshev polynomial of first kind $T_n(x) = \cos(n \arccos x), |x| \le 1$ and $\mathcal{N}$ the set of the natural numbers. Let the points $\xi_1, ..., \xi_n$ be arbitrary and $\mathcal{P}_k$ the space of all polynomials of degree $\le k$,then there exist weights $\lambda_1, ..., \lambda_n$ such that the numerical quadrature of the type

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} \lambda_i f(\xi_i) \tag{1}$$

is exact for $f \in \mathcal{P}_{n-1}$. But it is exact for $f \in \mathcal{P}_{2n-1}$ if the points $\xi_1, ..., \xi_n$ are the zeros of the Legendre polynomial of degree n. Moreover, there is no quadrature using a linear combination of n values of f such that Eq.(1) holds for all polynomials of degree $2n$. This classical result is the well-known Gauss-Legendre quadrature. Because of the above theorem of Gauss it is natural to ask whether the points $\xi_1, ..., \xi_n$ can be chosen so that quadrature rules of the form

$$\int_{-1}^{1} f(x)w(x)dx = \sum_{i=1}^{n}\sum_{j=0}^{2s} \lambda_{ij} f^{(j)}(\xi_i) \tag{2}$$

will be exact for all $f \in \mathcal{P}_{2(s+1)n-1}$,where $w(x)$ is a weight function. In his interesting paper [13],Turán showed that the answer is positive. Moreover,he showed that the n zeros $\xi_1, ..., \xi_n$ of the monic polynomials of degree n minimizing the expression

$$\int_{-1}^{1} |p(x)|^{2s+2} w(x)dx \tag{3}$$

over all such polynomials gives a quadrature of maximum degree of accuracy,

$$\int_{-1}^{1} f(x)w(x)dx = \sum_{i=1}^{n} \lambda_i f(\xi_i), \qquad f \in \mathcal{P}_{2(s+1)n-1}. \tag{4}$$

As Turán pointed out in [14],particularly interesting is the case when

$$w(x) = (1-x^2)^{-\frac{1}{2}}. \tag{5}$$

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In 1930, S.Bernstein [1] showed that $2^{1-n}T_n(x)$ minimizes all integrals of the type

$$\int_{-1}^{1} \frac{|p_n(x)|^k}{\sqrt{1-x^2}}dx, \qquad k \in \mathcal{N}. \tag{6}$$

So the Turán-Chebyshev formula

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \sum_{i=1}^{n}\sum_{j=0}^{2s} \lambda_{ij} f^{(j)}(x_{in}) \tag{7}$$

with $x_i = \cos\frac{(2i-1)\pi}{2n}, i = 1, ..., n$, is exact for $f \in \mathcal{P}_{2(s+1)n-1}$. Turán [14] has raised

Problem 26. Give an explicit formula for λ_{ij} and determine its asymptotic behavior as $n \to \infty$.

In this regard, Micchelli and Rivlin [6] have proved the following

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \frac{\pi}{n}\{\sum_{i=1}^{n} f(x_i) + \sum_{j=1}^{s} \frac{1}{2j4^{jn}}\binom{2j}{j} f'[x_1^{2j}, ..., x_n^{2j}]\}, \tag{8}$$

where $f[x_1^{2j}, ..., x_n^{2j}]$ designates the divided difference of the function f with each x_i repeated $2j$ times. For related work, see [5],[7]-[11] and references cited therein. Recently, Gori and Micchelli [3] considered the class $\mathcal{W}_n$ of weight functions to consist of all nonnegative integrable functions w on $[-1, 1]$ such that

$$w\sqrt{1-x^2} = \sum_{k=0}^{\infty}{}' \ \ \rho_k T_{2kn}(x), \tag{9}$$

where the prime on the summation indicates that the term corresponding to $k = 0$ is halved.

Accordingly, for every $w \in \mathcal{W}_n$ and $f \in C[-1, 1]$ we have

$$\int_{-1}^{1} f(x)w(x)dx = \frac{\pi}{2}\sum_{k=0}^{\infty}{}' \ \ \rho_k A_{2kn}(f), \tag{10}$$

where

$$A_n(f) = \frac{2}{\pi}\int_{-1}^{1} f(x)T_n(x)\frac{dx}{\sqrt{1-x^2}}. \tag{11}$$

Thus formula (10), and consequently (7), reduces to explicit expression for $A_{2kn}(f)$. Gori and Micchelli [3] obtained

Theorem A Let $j, k, s \in \mathcal{N}_0, \forall f \in \mathcal{P}_{2(s+1)n-1}$. Then

$$A_{2kn}(f) = \sum_{j=0}^{s} H_{kj} f'[x_1^{2j}, ..., x_n^{2j}], \tag{12}$$

where H_{kj} is implicitly defined by the following formal power series for $j, k \geq 1$, $|z| < 4^{n-1}$,

$$\sum_{j=1}^{\infty} H_{kj} j z^j = n^{-1}4^{(n-1)k}z^{-k}(1 - \sqrt{1 - 4^{-n+1}z})^{2k}(1 - 4^{-n+1}z)^{-\frac{1}{2}}, \tag{13}$$

for $k = 0, j \geq 1$,

$$\sum_{j=1}^{\infty} H_{0j} j z^j = n^{-1}((1 - 4^{-n+1}z)^{-\frac{1}{2}} - 1), \qquad |z| < 4^{n-1}, \tag{14}$$

$$H_{00} = \frac{2}{n}, \tag{15}$$

$$k \geq 1, H_{k0} = 0. \tag{16}$$

Theorem B Let $j, k, s \in \mathcal{N}_0, \forall f \in \mathcal{P}_{(2s+3)n-1}$,

$$A_{(2k+1)n}(f) = \frac{2}{n}\sum_{j=0}^{s} \hat{H}_{kj} f'[x_1^{2j+1}, ...x_n^{2j+1}], \tag{17}$$