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MATHEMATICAL ANALYSIS FOR QUADRILATERAL ROTATED $\mathcal{Q}_1$ ELEMENT III: THE EFFECT OF NUMERICAL INTEGRATION[*1)]

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Abstract

This is the third part of the paper for the rotated $\mathcal{Q}_1$ nonconforming element on quadrilateral meshes for general second order elliptic problems. Some optimal numerical formulas are presented and analyzed. The novelty is that it includes a formula with only two sampling points which excludes even a $\mathcal{Q}_1$ unisolvent set. It is the optimal numerical integration formula over a quadrilateral mesh with least sampling points up to now.

Key words: Quadrilateral rotated $\mathcal{Q}_1$ element, Numerical quadrature.

1. Numerical Integration Formulas

Throughout this paper, we adopt the notations appeared in [4]. Moreover, for any bounded domain D or its subdomain D_1, we denote $⨍_D f\,dx$ or $⨍_{D_1} f\,dx$ by the integral mean for any function $f \in L^1(D)$ or $f \in L^1(D_1)$.

We define the quadrature formulas on the reference square $\hat{K} = [-1,1] \times [-1,1]$ as follows:

$$\int_{\hat{K}} \hat{\phi}(\xi,\eta)d\xi d\eta \approx \sum_{i=1}^{I} \hat{\omega}_i \hat{\phi}(\hat{Q}_i), \qquad \hat{\phi} \in C(\hat{K}),$$

where the weight $\hat{\omega}_i > 0$, the quadrature point $\hat{Q}_i = (\xi_i, \eta_i) \in \hat{K},\ i = 1, \cdots, I$. Let $\hat{Q} = Span\{1, \xi, \eta, \xi^2 - \eta^2\}$, we assume that the quadrature is exact on $\hat{Q}$, hence it is also exact on $\mathcal{P}_1(\hat{K})$. The following four schemes will be considered:

$$\begin{array}{lll}
\text{Scheme1}: I = 4, & \hat{\omega}_i = 1, & \{\hat{Q}_i\}_{i=1}^4 = (-1,-1), (1,-1), (1,1), (-1,1),\\
\text{Scheme2}: I = 4, & \hat{\omega}_i = 1, & \{\hat{Q}_i\}_{i=1}^4 = (-1,0), (0,-1), (1,0), (0,1),\\
\text{Scheme3}: I = 3, & \hat{\omega}_i = 4/3, & \{\hat{Q}_i\}_{i=1}^4 = (-1,-1), (1,0), (0,1),\\
 & \hat{\omega}_i = 4/3, & \{\hat{Q}_i\}_{i=1}^4 = (1,-1), (-1,0), (0,1),\\
 & \hat{\omega}_i = 4/3, & \{\hat{Q}_i\}_{i=1}^4 = (1,1), (-1,0), (0,-1),\\
 & \hat{\omega}_i = 4/3, & \{\hat{Q}_i\}_{i=1}^4 = (-1,1), (1,0), (0,-1).\\
\text{Scheme4}: I = 2, & \hat{\omega}_i = 2, & \{\hat{Q}_i\}_{i=1}^2 = (-1,-1), (1,1), \text{ or } (1,-1), (-1,1).
\end{array}$$

In Figure 1, we only draw one case of Scheme 3 and Scheme 4, the other cases can be obtained symmetrically.

Remark 1.1. Unlike the standard quadrature formula, the above four formulas are not required to be exact either for the quadratic or for the bilinear polynomial space, but for the finite element space itself only. In particular, the scheme 4 does not contain even a $\mathcal{Q}_1(\hat{K})$-unisolvent set.

Remark 1.2. In fact, there are some other possibilities for obtaining a quadrature formula. For example, in the scheme 1, if we denote the weights $\hat{\omega}_i$ in the counterclockwise manner, then the following choices are also possible:

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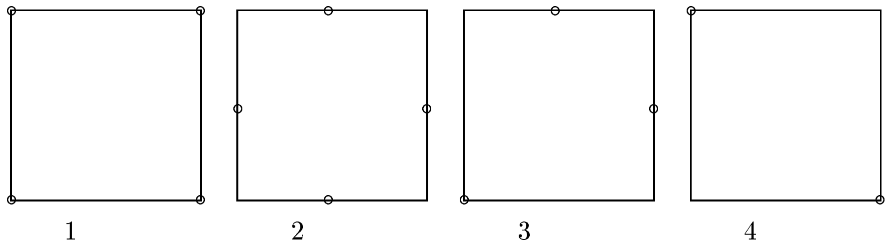

Figure 1

1. $\hat{\omega}_1 + \hat{\omega}_2 + \hat{\omega}_3 + \hat{\omega}_4 = 4$,
2. $\hat{\omega}_1 = \hat{\omega}_3$ and $\hat{\omega}_2 = \hat{\omega}_4$.

2. Analysis of Quadrature Formulas

The quadrature on an element K is given by

$$\int_K \phi dx \approx \sum_{i=1}^{I} \Omega_{i,K} \phi(Q_{i,K}) \equiv \mathcal{Q}_K(\phi),$$

where $\phi(x) = \hat{\phi}(\hat{x})$, $\omega_{i,K} = \hat{\omega}_i J_K(\hat{Q}_i)$, $Q_{i,K} = F_K(\hat{Q}_i)$. The quadrature error functional is denoted by

$$\begin{aligned} E_K(\phi) &= \int_K \phi(x)dx - \sum_{i=1}^{I} \omega_{i,K} \phi(Q_{i,K}), \\ \hat{E}_{\hat{K}}(\hat{\phi}) &= \int_{\hat{K}} \hat{\phi} - \sum_{i=1}^{I} \hat{\omega}_i \hat{\phi}(\hat{Q}_i), \end{aligned}$$

where $E_K(\phi) = \hat{E}_{\hat{K}}(\hat{\phi} J_K)$. Now we apply the quadrature formula $\mathcal{Q}_K$ to the finite element equation (2.11) in [4]. Define

$$a_h(u,v) \equiv \sum_{K \in \mathcal{T}_h} \mathcal{Q}_K[a_{11}\partial_x u \partial_x v + a_{12}(\partial_x u \partial_y v + \partial_y u \partial_x v) + a_{22}\partial_y u \partial_y v + auv],$$

and $(f,v)_h \equiv \sum_{K \in \mathcal{T}_h} \mathcal{Q}_K(fv)$, we solve the following equation:

$$a_h(u_h, v) = (f, v)_h \quad \forall v \in V_{0,h}. \tag{2.1}$$

From now on, we always assume that the *Bi-Section Condition* [7] holds.

Theorem 2.1. *Suppose $a_{ij}, a \in W^{1,\infty}(\Omega), f \in W^{1,q}(\Omega), q > 2$, and $u \in H_0^1(\Omega), u_h \in V_{0,h}$ are the solution of (1.1) in [4] and (2.1), respectively, then*

$$|u - u_h|_h \le Ch \left(\sum_{i,j=1}^{2} (\|a_{ij}\|_{1,\infty} + \|a\|_{1,\infty})\|u\|_2 + \|f\|_{1,q} \right).$$

This theorem is a direct consequence of following lemmas.

Lemma 2.1. *The modified bilinear form $a_h(\cdot,\cdot)$ with the quadrature $\mathcal{Q}_K$ is V_h-ellipticity, that is*

$$a_h(v_h, v_h) \ge C\|v_h\|_h^2 \quad \forall v_h \in V_{0,h}. \tag{2.2}$$