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ON KORN'S INEQUALITY*1)

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Abstract

This paper is devoted to give a new proof of Korn's inequality in L^r-norm $(1 < r < \infty)$.

Key words: Korn's inequality.

1. Introduction

Korn's inequality is fundamental in the theory and the numerical analysis for the elasticity. There have been many nice proofs of Korn's inequality in the literatures(see [4] and the references therein). The work [5] proposed an intuitive exposition and heuristic proof of Korn's inequality. And the works [2] and [7] give an interesting result, which is useful tool in the proof of Korn's inequality, as for example in the works [3], [6].

In this paper, we intend to show a new proof of Korn's inequality in L^r-norm $(1 < r < \infty)$, in the plane, with the help of the heurisitc work [4] and the result of [2], [7].

2. Notation and Preliminaries

We begin with some notation. Let $\Omega \subset R^n (n = 2, 3)$ denote the bounded domain with smoothly boundary $\partial\Omega$ or the polygon. Let $\vec{v}$ be the n-dimensional vector valued function defined in Ω, and

$$\epsilon_{ij}(\vec{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j), \quad \partial_j v_i = \frac{\partial v_i}{\partial x_j}, \quad 1 \le i,\ j \le n. \tag{2.1}$$

And in this paper, the notation in Sobolev spaces [1] will be used.

Korn's inequality, in L^2 version, can be stated as follows: There exists C=Const. > 0, such that

$$\sum_{i,j} \|\epsilon_{ij}(\vec{v})\|^2_{0,\Omega} + \|\vec{v}\|^2_{0,\Omega} \ge C\|\vec{v}\|^2_{1,\Omega} \quad \forall \mathbf{v} \in E, \tag{2.2}$$

where

$$E = \{\vec{w} \in (L^2(\Omega))^2 : \epsilon_{ij}(\vec{w}) \in L^2(\Omega) \forall i, j\}. \tag{2.3}$$

Korn's inequality (2.2) means that the following cotaining relationship holds:

$$E \subset (H^1(\Omega))^n. \tag{2.4}$$

The relation (2.4) seems to be unexpected at the first glance, because, for the case $n = 3$, only six independent linear combinations of partial derivatives of $\vec{v} \in (H^1(\Omega))^3$ belong to $L^2(\Omega)$. However when we consider it in depth, as in [5], we find that all second order partial derivatives of $\vec{v}$ can be presented by the partial derivatives of $\epsilon_{ij}(\vec{v})$:

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_j}\epsilon_{ik}(\vec{v}) + \frac{\partial}{\partial x_k}\epsilon_{ij}(\vec{v}) - \frac{\partial}{\partial x_i}\epsilon_{jk}(\vec{v}). \tag{2.5}$$

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Thus if $\vec{v} \in E$, then

$$\frac{\partial}{\partial x_k}\Big(\frac{\partial v_i}{\partial x_j}\Big) \in H^{-1}(\Omega)\ \forall\ i,j,k, \tag{2.6}$$

which, roughly speaking, can be seen (the rigorous proof can be found in [5], added by $\partial v_i/\partial x_j \in H^{-1}(\Omega)\forall i,j$) as

$$\frac{\partial v_i}{\partial x_j} \in L^2(\Omega)\ \forall\ i,j. \tag{2.7}$$

This means that $\mathbf{v} \in (H^1(\Omega))^n$.

3. The Proof of Korn's Inequality

In this section, we present a new proof of Korn's inequality in L^r version, $1 < r < \infty$, in the plane ($n = 2$), which can be stated in the following:

Theorem 1 (Korn's Inequality). *There exists a positive constant α, such that*

$$\sum_{i,j} \|\epsilon_{ij}(\vec{v})\|_{0,r,\Omega} + \|\vec{v}\|_{0,r,\Omega} \geq \alpha\|\vec{v}\|_{1,r,\Omega} \quad \forall\ \vec{v} \in (W^{1,r}(\Omega))^2. \tag{3.1}$$

In order to prove Theorem 1, we need some lemmas.

Lemma 1. *For all $w \in L^r(\Omega)$,*

$$\begin{cases} \|w\|_{-1,r,\Omega} \leq \|w\|_{0,r,\Omega}, \\ \|\nabla w\|_{-1,r,\Omega} \leq \|w\|_{0,r,\Omega}. \end{cases} \tag{3.2}$$

Lemma 1 can be proved easily by the definition of the $W^{-1,r}(\Omega)-$norm.

Lemma 2 (c.f.[2],[6]). *Assume that $\Omega \subset R^2$ be a bounded smoothly domain or polygon. Let*

$$L_0^r(\Omega) = \{p \in L^r(\Omega) : \int_\Omega p dx = 0\}. \tag{3.3}$$

Then for any given $p \in L_0^{r'}(\Omega),\ 1 < r' = r/(r-1) < \infty$, r' the conjugate number of r, there exists $\vec{\phi}_0 \in (W_0^{1,r'}(\Omega))^2$, such that

$$div\vec{\phi}_0 = p\ in\ \Omega, \quad \|\vec{\phi}_0\|_{1,r',\Omega} \leq C\|p\|_{0,r',\Omega}, \tag{3.4}$$

with a constant C independent of $\vec{\phi}_0$ and p.

Lemma 3. *For every function $v \in L^r(\Omega),\ 1 < r < \infty$,*

$$\|v\|_{0,r,\Omega} \leq \frac{1}{2C}\|\nabla v\|_{-1,r,\Omega} + \frac{1}{|\Omega|^{1/r'}}\Big|\int_\Omega v dx\Big|, \tag{3.5}$$

with the same $C = Const.$ as in (3.4) and $|\Omega| = \int_\Omega 1 dx$.

Proof. For any given $v \in L^r(\Omega)$ let

$$\hat{v} = v - \frac{1}{|\Omega|}\int_\Omega v dx,$$

then

$$\|v\|_{0,r,\Omega} \leq \|\hat{v}\|_{0,r,\Omega} + \frac{1}{|\Omega|^{1/r'}}\Big|\int_\Omega v dx\Big|. \tag{3.6}$$

And

$$\|\hat{v}\|_{0,r,\Omega} = \sup_{w\in L^{r'}(\Omega)} \frac{\int_\Omega \hat{v}\cdot w dx}{\|w\|_{0,r',\Omega}} = \sup_{w\in L^{r'}(\Omega)} \frac{\int_\Omega \hat{v}(\hat{w} + \frac{1}{\Omega}\int_\Omega w dy)dx}{\|w\|_{0,r',\Omega}} \leq \frac{1}{2}\sup_{\hat{w}\in L_0^{r'}(\Omega)} \frac{\int_\Omega \hat{v}\cdot\hat{w}dx}{\|\hat{w}\|_{0,r',\Omega}},$$