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A COMBINED HYBRID FINITE ELEMENT METHOD FOR PLATE BENDING PROBLEMS*[1)]

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Abstract

In this paper, a combined hybrid method is applied to finite element discretization of plate bending problems. It is shown that the resultant schemes are stabilized, i.e., the convergence of the schemes is independent of inf-sup conditions and any other patch test. Based on this, two new series of plate elements are proposed.

Key words: Combined hybrid finite element, Weakly compatible.

1. Introduction

The success of finite element methods for the numerical solution of boundary value problems for elliptic partial differential equations is, to a large extent, due to the variational principles upon which these methods are built. The assumed stress hybrid methods pioneered by Pian and Tong (see[8],[10],etc.) are based on modified complementary energy principles and proved to be very successful in a number of applications; see, e.g.,[3],[7],[8],[10],[11],[19], etc.. For 4th-order problems, the hybrid methods can relax the C^1-continuity for deflection elements so that sufficient flexibility in the finite element solution can be gained. However, because of the "saddle-point" nature of the hybrid models, some strict stability conditions such as inf-sup conditions or LBB conditions must be satisfied by deflection and bending moments (e.g.,[3]), and then the formulations of hybrid elements can not yet be simplified to a degree comparable to the use of shape function routine in the conventional displacement methods. And due to the complicated self-equilibrium equations, application of assumed stress hybrid elements to shell analysis is not convenient.

To avoid the inf-sup difficulties, the least-squares method (see,e.g.,[1] and the references therein) having developed in the past decade seems to be an efficient way. But as pointed out in [1], for 4th-order problems, some conforming shape functions are required and the resulting least-squares finite element method also fails to be practical because the condition numbers of the corresponding discrete problems are $O(h^{-4})$ compared with the $O(h^{-2})$ condition numbers that result from standard Galerkin methods for the same problem.

Recently, a new hybrid finite element method for linear elasticity problems named as combined hybrid method was suggested by Zhou (see [23],[26]). This approch is based on a so-called combined variational principle, i.e., a homotopy family of optimization conditions of two dual systems of saddle point problem–one is the domain-decomposed Hellinger-Reissner principle, the other is the primal hybrid variational principle, a dual to the former. Theoritical analyses[23] and numerical tests[26] both showed that the combined hybrid method possesses not only the features of hybrid methods, but also almost all the significant and valuable properties of the least-squres methods, such as: it can circumvent the inf-sup conditions, and then the

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weak problems are in general coercive; the resulting algebraic problems are symmetric and positive definite; essential boundary conditions can be imposed in a weak sense; and finite element spaces for displacement and stress can be chosen independently, etc..

In this paper, the combined variational principle, as a rational approach to incompatible displacement schemes, is applied to finite element discretization of plate bending problems. It is shown that the resultant schemes, named as combined hybrid finite element schemes, is stabilized, i.e., the convergence of the new incompatible element schemes is independent of inf-sup conditions and any other patch test. Then the deflection and the bending moments subspace can be chosen independently. The C^1-continuity for deflection interpolations is relaxed. The self-equilibrium constraint on the bending moments subspace $\mathbf{V}^h$ is not required.Two new series of plate elements to are given to show another feature of the combined hybrid method, i.e., in building the plate bending finite elements there exists great possibility in the choice of stress/strain-enriched interpolations to enhance accurary of the schemes.

The paper is arranged as follows. In section 2 the combined variational principle is derived and then a mathematical foundation of the stabilized hybrid method is established. Section 3 is devoted to the discussion of stabilized hybrid schemes and convergence. The error estimates are deduced. Finally, two new series of plate elements are given in section 4.

In what follows the letter C will represent different constant independent of the mesh size h at its each occurence.

2. Combined Variational Principle

We consider the following plate bending problem:

$$\begin{cases} \mathbf{divdiv}\sigma = f, & in\ \Omega, \\ \sigma = m(\mathbf{D}_2 u), & in\ \Omega, \\ u = \nabla u \cdot n = 0, & on\ \Gamma = \partial\Omega. \end{cases} \tag{2.1}$$

where $\Omega \subset \Re^2$ is a bounded open set, u represents vertical deflection, σ the bending moments, and n the outer normal unit vector along Γ. The operators $\mathbf{divdiv}, \mathbf{D}_2$ and m are defined respectively as follows:

$$\mathbf{divdiv}\tau = \partial_{11}\tau_{11} + 2\partial_{12}\tau_{12} + \partial_{22}\tau_{22},$$

$$\mathbf{D}_2 v = \begin{pmatrix} \partial_{11}v & \partial_{12}v \\ \partial_{12}v & \partial_{22}v \end{pmatrix},$$

$$m(\tau) = \begin{pmatrix} \tau_{11} + \nu\tau_{22} & (1-\nu)\tau_{12} \\ (1-\nu)\tau_{12} & \nu\tau_{11} + \tau_{22} \end{pmatrix}$$

for any symmetric tensor τ, and $\nu \in (0, 0.5)$ denotes the Poisson's coefficient, $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, $i, j = 1, 2$.

We know that for this problem the two basic solution spaces are the deflection space $H_0^2(\Omega)$ and the bending moments space $H(\mathbf{divdiv}; \Omega) := \{\tau \in (L^2(\Omega))_s^4; \mathbf{divdiv}\tau \in L^2(\Omega)\}$, where $(L^2(\Omega))_s^4$ is the space of square integrable 2×2 symmetric tensors.

To relax continuity, we introduce the following two piecewise Sobolev spaces to replace $H_0^2(\Omega)$ and $H(divdiv; \Omega)$:

$$\mathbf{V} := \prod_{K \in T_h} H(\mathbf{divdiv}; K),$$

$$U := \{v \in \prod_{K \in T_h} H^2(K); u = \nabla u \cdot n = 0, on\ \Gamma\},$$

where $T_h = \{K\}$ denotes a regular subdivision of Ω, with mesh diameter h_K for any $K \in T_h$. We also need the following Lagrange multiplier space as

$$U_c := H_0^2(\Omega) / \prod_{K \in T_h} H_0^2(K).$$