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REAL PIECEWISE ALGEBRAIC VARIETY *1)

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Abstract

We give definitions of real piecewise algebraic variety and its dimension. By using the techniques of real radical ideal, P-radical ideal, affine Hilbert polynomial, *Bernstein-net form* of polynomials on simplex, and decomposition of semi-algebraic set, etc., we deal with the dimension of the real piecewise algebraic variety and real Nullstellensatz in C^μ spline ring.

Key words: Dimension, Real piecewise algebraic variety, C^μ spline ring.

1. Introduction

Multivariate splines as piecewise polynomials have been studied intensively in the past 20 years, and have become a kind of fundamental tool for computational geometry, numerial analysis, approximation, and optimization, etc.([8]). The interpolation of scattered date by multivariate splines is an important topic in computational geometry. It is concerned with several practical areas such as CAD,CAM, CAE, and Image processing. However, the construction of explicit interpolation schemes (especially Lagrange interpolation schemes) for spline spaces on given partition leads to complex problems. In principle, to solve an interpolation problem, one has to deal with the properties of piecewise algebraic variety and piecewise algebraic curves. Piecewise algebraic variety is the set of common zeros of the multivariate splines. Therefore, a key problem on interpolation by multivariate splines is to study the piecewise algebraic variety and piecewise algebraic curves. Piecewise algebraic variety, as a generalization of algebraic variety, is a new and important concept in algebraic geometry and computational geometry, and has great significance in theory and application (cf [8]). Some fundamental properties of piecewise algebraic variety were given in ([8],[9]). A generalization of Bezout theorem of piecewise algebraic curves has been obtained in ([10]).

In this paper, we give definitions of real piecewise algebraic variety and its dimension. By applying the techniques of real radical ideal, P-radical ideal(P be a cone), affine Hilbert polynomial, *Bernstein-net form* of polynomials on simplex, decomposition of semi-algebraic set, etc, ([1],[3],[5],[6]), we deal with the dimension of real piecewise algebraic variety and real Nullstellensatz in C^μ spline ring.

2. Definitions and Preliminaries

Let R be the real number field. We define n-space over R, denoted by R^n, to be the set of all n-tuples of R. An element $x = (x_1, \dots, x_n) \in R^n$ will be called a point. Denote by $R[x]$ the polynomial ring in n variable over R.

By finite hyperplane patches in R^n, we subdivide a simply connected basic closed semi-algebraic domain $D \subseteq R^n$ with dimension n into finite simply connected subdomains with dimensions n(cf[5]), and each of them is homeomorphic to a hypercube, which is called a

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cell. So we get a partition of the domain D. Denote by Δ the partition of the domain D which consists of all partition cells $\delta_1, \ldots, \delta_T$ and their faces $S_1, \ldots, S_E$. Where $S_1, \ldots, S_E$ are algebraic hypersurface patches or algebraic varieties with dimensions $< n$. The face of each cell $\delta_i \in \Delta$ consists of finite partition faces. It is well known that every cell δ_i can be written as the intersection of a collection of affine halfspaces, namely,

$$\delta_i = \{x \in R^n |\ S_1^{(i)}(x) \geq 0, \ldots, S_{q_i}^{(i)}(x) \geq 0, \quad S_\alpha^{(i)} \in R[x], \alpha = 1, \quad \ldots, q_i\}, \qquad 1^*$$
$$i = 1, 2, \ldots, T.$$

Denote by $P(\Delta)$ the collection of functions f on D such that for every cell δ_i the restriction of f on δ_i, $f|_{\delta_i}$, is a polynomial function. $f|_{\delta_i}$ refers also to a polynomial corresponding to f on cell δ_i if no confusion can arise, i.e., $f|_{\delta_i} \in R[x]$. It is obvious that

$$S^\mu(\Delta) = \{f |\ f \in C^\mu(D) \cap P(\Delta)\}, \quad \mu \geq 0$$

is a ring over R, which is called C^μ spline ring. It is clear $R[x] \subseteq S^\mu(\Delta)$. The degree of $f \in S^\mu(\Delta)$ denoted by $degf$ is the maximal degree of polynomials corresponding to f on all cells of Δ. We say that

$$S_m^\mu(\Delta) := \{f |\ degf \leq m, \quad f \in S^\mu(\Delta)\}$$

is a multivariate spline space with degree m and smoothness μ. $S_m^\mu(\Delta)$ is a finite dimensional linear vector space on $m, \mu \geq 0$.

Now we discuss real C^μ piecewise algebraic variety on a partition Δ of a domain D in R^n.

Definition 2.1. *Let Δ be a partition of a domain $D \subseteq R^n$, Denote by δ_i, $i = 1, \ldots, T$, all the cells of Δ. If there exist $f_1, \ldots, f_s \in S^\mu(\Delta)$, $\mu \geq 0$ such that*

$$\mathcal{Z} = \mathcal{Z}(f_1, \ldots, f_s) = \{x \in D : \quad f_i(x) = 0, \quad i = 1, \ldots, s\},$$

then $\mathcal{Z}$ is called a **real C^μ piecewise algebraic variety** *defined by $f_1 \ldots, f_s$ on the partition Δ.*

Thus a real C^μ piecewise algebraic variety $\mathcal{Z} = \mathcal{Z}(f_1, \ldots, f_s)$ is the set of all real solutions of the system of equations $f_1 = \ldots = f_s = 0$, i.e., the set of all real common zeros of C^μ multivariate splines $f_1, \ldots, f_s$.

Let J be an ideal of $S^\mu(\Delta)$. Denote by $\mathcal{Z}(J)$ the set

$$\mathcal{Z}(J) = \{x \in D : \quad f(x) = 0, \quad f \in J\}.$$

Since $S^\mu(\Delta)$ is a Nöther ring([8]), J has a finite set of generators $f_1, ..., f_s \in S^\mu(\Delta)$ such that

$$\mathcal{Z}(J) = \mathcal{Z}(f_1, \ldots, f_s).$$

Thus $\mathcal{Z}(J)$ is a real C^μ piecewise algebraic variety on the partition Δ of the domain D.

If J is an ideal of $R[x]$, then

$$\mathcal{Z}(J) = \{x \in R^n | f(x) = 0, \quad f \in J\}$$

is a real algebraic variety.

Let $V \subset R^n$ be a real algebraic variety. Denote by $\mathcal{P}(V) = R[x_1, ..., x_n]/\mathcal{I}(V)$ the ring of polynomial functions on V, where

$$\mathcal{I}(V) = \{f(x) \in R[x] \ : f(a) = 0, \quad a \in V\}.$$

Denote by $dim(V)$ the dimension of V. $dim(V)$ is equal to the dimension of the ring $\mathcal{P}(V)$, i.e., the maximal length of chains of prime ideal of $\mathcal{P}(V)$([4],[5],[7]).

Let $\mathcal{Z}$ be a real C^μ piecewise algebraic variety. $clos_{zar}(\mathcal{Z} \cap \delta_i)$ denotes the Zariski closure of $\mathcal{Z} \cap \delta_i, i = 1, \ldots, T$. If for some $\mathcal{Z} \cap \delta_i = \emptyset$, then $dim(clos_{zar}(\mathcal{Z} \cap \delta_i)) = -1$.(cf. [4],[7])