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ON THE ERROR ESTIMATE OF NONCONFORMING FINITE ELEMENT APPROXIMATION TO THE OBSTACLE PROBLEM*1)

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Abstract

This paper is devoted to analysis of the nonconforming element approximation to the obstacle problem, and improvement and correction of the results in [11], [12].

Key words: Obstacle problem, Nonconforming finite element.

1. Introduction

For the conforming (i.e. C^0-) linear finite element approximation to the obstacle problem, the error bound $O(h)$ has been obtained by Falk [5] in homogeneous boundary data and Brezzi et.al., [3] in nonhomogeneous data and with lower order term. The author considered nonconforming (i.e. nonC^0-) finite element approximation to the obstacle problem in [10] and [11]. Later, [12] presented a rigorous proof of the error bound $O(h)$ and correction of the proof in [11] for nonconforming linear element approximation to the obstacle problem under the hypothesis that the free boundary has finite length as in [3].

In general, the length of the free boundary could be not finite, because there exist probably infinite simply connected coincidence sets, while the length of the boundary of each such coincidence set is finite for the smooth solution of the obstacle problem. In fact, one can constructe example of infinite simply connected sets in a bounded domain, with property that the total length of the boundaries of all the sets is infinite. Thus it makes sense to estimate the error bound of nonconforming linear element approximation to the obstacle problem without the hypothesis of finite length of the free boundary.

In this paper, by the similar way as [3], we obtaine the error bound $O(h)$ for the nonconforming linear element approximation to the obstacle problem without the hypothesis of finite length of the free boundary. And in [10], [11] the author also analyzed Wilson's element for the obstacle problem with an unnatural construction of the discrete convex set K_h. In this paper we consider Wilson's element approximation to the obstacle problem with a natural and simple construction of the discrete convex set K_h, and obtaine the same error bound $O(h)$ as in [10], [11].

Let Ω be a bounded convex domain in R^2 with smooth boundary $\partial\Omega$, and $f \in L^2(\Omega), \chi \in H^2(\Omega), g$ be the trace of a function in $H^2(\Omega)$ on $\partial\Omega$ and $\chi \leq g$ on $\partial\Omega$. Let us consider the following obstacle problem:

$$\begin{cases} to \;\; find \quad u \in K, \;\; such \;\; that \\ a(u, v-u) \geq (f, v-u) \qquad \forall \;\; v \in K, \end{cases} \tag{1.1}$$

where

$$K = \{v \in H^1(\Omega) : v \geq \chi \;\; a.e.in \;\; \Omega, v = g \;\; on \;\; \partial\Omega\}, \tag{1.2}$$

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$$a(u,v)=\int_\Omega \nabla u\cdot\nabla v dx, \qquad (f,v)=\int_\Omega f\cdot v dx. \tag{1.3}$$

It is well known that (see [4]) problem (1.1) is equivalent to the following differential problem:

$$\begin{cases} -\Delta u = f \quad in \quad \Omega^+=\{x\in\Omega : u(x)>\chi(x)\}, \\ -\Delta u \geq f \quad in \quad \Omega^0=\{x\in\Omega : u(x)=\chi(x)\}, \\ u\geq \chi \quad in \quad \Omega, \quad and \quad u=g \quad on \quad \partial\Omega. \end{cases} \tag{1.4}$$

For the regularity of the solution of the obstacle problem (1.1), we now present a very impotant result by Brezis:

Lemma 1.1 (see [6], [7]). *If $f\in L^\infty(\Omega)\cap BV(\Omega),(g,\chi)\in C^3(\bar\Omega)$ with $\chi\leq g$ on $\partial\Omega$ and $\partial\Omega$ is sufficiently smooth, then the problem (1.1) has a solution*

$$u\in W^{s,p}(\Omega), \qquad 1<p<\infty, \quad s<2+\frac{1}{p}. \tag{1.5}$$

We now consider the finite element approximation to problem (1.1). Let $\mathcal{T}_h$ be a regular subdivision on Ω, $\Omega_h=\cup_{T\in\mathcal{T}_h}T$, with $T\in\mathcal{T}_h$ denoting the element, and let $V_h\subset L^2(\Omega_h)$ be the finite element space with norm $\|\cdot\|_h$, and $K_h\subset V_h$ be a closed convex subset as an approximation of K. Then the finite element approximate problem of (1.1) is the following:

$$\begin{cases} to \quad find \qquad u_h\in K_h, \qquad such\ that \\ a_h(u_h,v_h-u_h)\geq(f,v_h-u_h) \qquad \forall \quad v_h\in K_h. \end{cases} \tag{1.6}$$

where

$$a_h(u_h,v_h)=\sum_{T\in\mathcal{T}_h}\int_T \nabla u_h\cdot\nabla v_h dx. \tag{1.7}$$

2. The Nonconforming Linear Element Approximation

Let $\mathcal{T}_h$ be a regular triangulation of Ω, the vertices of the element T be denoted by $a_i^T, 1\leq i\leq 3$, and the midpoints of the edges of the element T be denoted by $m_i^T, 1\leq i\leq 3$. And let X_h denote the nonconforming linear element space with respect to the triangulation $\mathcal{T}_h$. Let (see Fig.2.1)

$$V_h=\{v_h\in X_h : v_h(m)=g(P_m) \ \ \forall \ \ nodes \ \ m\in\partial\Omega_h\}, \tag{2.1}$$

$$K_h=\{v_h\in V_h : v_h(m_i^T)\geq\chi(m_i^T) \ \ \forall \ \ T\in\mathcal{T}_h \ \ and \ \ m_i^T\notin\partial\Omega_h\}. \tag{2.2}$$

Let

$$V_h^0=\{v_h\in X_h : v_h(m)=0 \ \ \forall \ \ nodes \ \ m\in\partial\Omega_h\}, \tag{2.3}$$

then

$$\|w_h\|_h=a_h(w_h,w_h)^{\frac{1}{2}} \tag{2.4}$$

is a norm in V_h^0.

In order to estimate the error bound of the approximate problem (1.6), firstly we have the abstract error estimate:

Lamma 2.1. *Assume that u and u_h denote the solutions of the problems (1.1) and (1.6) respectively. Then $\forall v_h\in K_h$, the following inequalities hold:*

$$\|u-u_h\|_h\leq\|u-v_h\|_h+\|v_h-u_h\|_h, \tag{2.5}$$