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FRACTIONAL DIFFUSION EQUATIONS WITH INTERNAL DEGREES OF FREEDOM [*1)]

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Abstract

We present a generalization of the linear one-dimensional diffusion equation by combining the fractional derivatives and the internal degrees of freedom. The solutions are constructed from those of the scalar fractional diffusion equation. We analyze the interpolation between the standard diffusion and wave equations defined by the fractional derivatives. Our main result is that we can define a diffusion process depending on the internal degrees of freedom associated to the system.

Key words: Fractional derivative, Diffunors, Diffusion process, Generalized Dirac equation.

1. Introduction

It is well known the approach of Dirac to obtain his famous equation from the Klein-Gordon equation [1]. The free Dirac equation can be considered as the root square of the Klein-Gordon equation. In a more general context Morinaga and Nono [2] analyzed the integer s-root of the partial differential equations of the form

$$\sum_{|I|=s} a_I \frac{\partial^{|I|}}{\partial x^I} \phi = \phi \tag{1}$$

The $s-$root is the first order system

$$\sum_{i=1}^{n} \alpha_i \frac{\partial \Phi}{\partial x_i} = \Phi \tag{2}$$

being $\alpha_1, ...\alpha_n$ matrices. From the physical point of view the α_k describe internal degrees of freedom of the associated system.

The purpose of the paper is to generalize the above study to the case of fractional derivatives. In this context, we will consider the fractional diffusion equations with internal degrees of freedom obtained by generalization of the s-roots of the standard scalar linear diffusion equation. Thus, it is natural to consider the space and time fractional derivatives in a symmetric way through the framework of the standard Fourier transform

$$\frac{\partial^\alpha u(s)}{\partial s^\alpha} \longrightarrow (-i\kappa)^\alpha \hat{u}(\kappa) \tag{3}$$

being $\kappa \in \Re$.

In a future work we will consider the different following definitions of time and space fractional derivatives [3] which appear in other contexts:

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- *The time fractional derivative* [4] of order $\alpha > 0$ for a sufficiently well-behaved *causal* function $u(t)$ is defined as follows

$$\frac{d^\alpha}{dt^\alpha}u(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{u^{(m)}}{(t-\tau)^{\alpha+1-m}}\, d\tau \tag{4}$$

where $m = 1, 2, ...,$ and $0 \leq m-1 < \alpha < m$. This definition requires the absolute integrability of the derivative of order m.

- *The symmetric space fractional derivative* [5] of order $\alpha > 0$ of a sufficiently well-behaved function $u(x), x \in \Re$, is defined as the pseudo-differential operator characterized in its Fourier representation by

$$\frac{d^\alpha}{d \mid x \mid^\alpha}u(x) \longrightarrow - \mid \kappa \mid^\alpha \hat{u}(\kappa) \tag{5}$$

as before being $\kappa \in \Re$.

2. The Square Root of the Standard Linear Diffusion Equation

A possible definition of the root-square of the standard diffusion equation (SDE) in one space dimension, $u_t - u_{xx} = 0$, is the following:

$$(A\frac{\partial^{1/2}}{\partial t^{1/2}} + B\frac{\partial}{\partial x})\psi(x,t) = 0 \tag{6}$$

where A and B are matrices satisfying the conditions:

$$A^2 = I \quad , \quad B^2 = -I \tag{7}$$

$$\{A, B\} \equiv AB + BA = 0 \tag{8}$$

being $\psi(x,t)$ multidimensional with at least two scalar space-time components. Also, every scalar component satisfies the SDE. Such solutions can be interpreted as *probability distributions with internal structure associated to internal degrees of freedom of the system.* We could name them **diffunors** in analogy with the spinors in Quantum Mechanics.

We have two possible realizations of the above algebra in terms of real matrices 2×2 associated to the Pauli matrices:

- $$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{9}$$

- $$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{10}$$

Other realizations involving complex bidimensional matrices are possible, but taking into account the reference to the diffusion equation we only consider the real representations.

The solutions of (6) are related to the SDE in a simple way. As a an illustration, let us consider the representation (9), thus $\psi(x,t) = \begin{pmatrix} \varphi(x,t) \\ \chi(x,t) \end{pmatrix}$ such that $\chi(x,t) = \pm\varphi(x,t)$. We have two general independent solutions of (6):

$$\varphi(x,t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad and \quad \varphi(x,t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{11}$$