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A MODIFIED VARIABLE-PENALTY ALTERNATING DIRECTIONS METHOD FOR MONOTONE VARIATIONAL INEQUALITIES [*1)]

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Abstract

Alternating directions method is one of the approaches for solving linearly constrained separate monotone variational inequalities. Experience on applications has shown that the number of iteration significantly depends on the penalty for the system of linearly constrained equations and therefore the method with variable penalties is advantageous in practice. In this paper, we extend the Kontogiorgis and Meyer method [12] by removing the monotonicity assumption on the variable penalty matrices. Moreover, we introduce a self-adaptive rule that leads the method to be more efficient and insensitive for various initial penalties. Numerical results for a class of Fermat-Weber problems show that the modified method and its self-adaptive technique are proper and necessary in practice.

Key words: Monotone variational inequalities, Alternating directions method, Fermat-Weber problem.

1. Introduction

The mathematical form of variational inequalities consists of finding a vector $u^* \in \Omega$ such that

$$\text{VI}(\Omega, F) \qquad (u-u^*)^T F(u^*) \geq 0, \qquad \forall\ u \in \Omega, \tag{1}$$

where Ω is a nonempty, closed convex subset of $\mathcal{R}^l$, F is a continuous mapping from $\mathcal{R}^l$ to itself. In practice, many VI problems have the following separable structure, namely (e.g., [14]),

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad F(u) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix}, \tag{2}$$

$$\Omega = \{(x,y) | x \in \mathcal{X}, y \in \mathcal{Y}, Ax + By = b\}, \tag{3}$$

where $\mathcal{X} \subset \mathcal{R}^n$ and $\mathcal{Y} \subset \mathcal{R}^m$ are given closed convex sets, $f: \mathcal{X} \to \mathcal{R}^n$, $g: \mathcal{Y} \to \mathcal{R}^m$ are given monotone operators, $A \in \mathcal{R}^{r\times n}, B \in \mathcal{R}^{r\times m}$ are given matrices, and $b \in \mathcal{R}^r$ is a given vector.

By attaching a Lagrange multiplier vector $\lambda \in \mathcal{R}^r$ to the linear constraints $Ax + By = b$, the problem under consideration can be explained as a *mixed variational inequality* (VI with equality restriction $Ax + By = b$ and unrestricted variable λ):

$$\text{Find} \quad w^* \in \mathcal{W}, \quad \text{such that} \quad (w-w^*)^T Q(w^*) \geq 0, \quad \forall\, w \in \mathcal{W}, \tag{4}$$

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where

$$w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, \qquad Q(w) = \begin{pmatrix} f(x) - A^T\lambda \\ g(y) - B^T\lambda \\ Ax + By - b \end{pmatrix}, \qquad \mathcal{W} = \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^r. \tag{5}$$

Problem (4)-(5) is denoted as MVI($\mathcal{W}, Q$) and will be concerned in this paper. It has been well known (e.g., see [14]) that solving MVI($\mathcal{W}, Q$) is equivalent to finding a zero point of

$$e(w) := w - P_{\mathcal{W}}[w - Q(w)], \tag{6}$$

where $P_{\mathcal{W}}(\cdot)$ denotes the projection on $\mathcal{W}$. $\|e(w)\|$ can be viewed as a '*error bound*' that measures how much w fails to be a solution of MVI($\mathcal{W}, Q$).

As a tool for solving MVI($\mathcal{W}, Q$) problems, the alternating directions method was originally proposed by Gabay [5] and Gabay and Mercier [4]. At each iteration of this method, the new iterate $w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1}) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^r$ is generated from a given triple $w^k = (x^k, y^k, \lambda^k) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^r$ by the following procedure: First, x^{k+1} is obtained (with y^k and λ^k held fixed) by solving

$$(x' - x^{k+1})^T \Big(f(x^{k+1}) - A^T[\lambda^k - \beta(Ax^{k+1} + By^k - b)] \Big) \geq 0, \qquad \forall\ x' \in \mathcal{X}, \tag{7}$$

and then y^{k+1} is produced (with x^{k+1} and λ^k held fixed) by solving

$$(y' - y^{k+1})^T \Big(g(y^{k+1}) - B^T[\lambda^k - \beta(Ax^{k+1} + By^{k+1} - b)] \Big) \geq 0, \qquad \forall\ y' \in \mathcal{Y}. \tag{8}$$

Finally, the multipliers are updated by

$$\lambda^{k+1} = \lambda^k - \gamma\beta(Ax^{k+1} + By^{k+1} - b), \tag{9}$$

where $\gamma \in (0, \frac{1+\sqrt{5}}{2})$ and $\beta > 0$ are given constants. This method is referred to as a *method of multiplier* in the literature [5], and the convergence proof can be found in [4, 6] (for $B = I$) and [13, 15] (for general B). Further studies and applications of such methods can be found in Glowinski [6], Glowinski and Le Tallec [7], Eckstein and Fukushima [1] and He and Yang [9].

Experience on applications [2, 3, 12] has shown that if the fixed penalty β is chosen too small or too large the solution time can significantly increase. In order to improve such methods, recently, Kontogiorgis and Meyer [12] presented a more general alternating directions method, in which they took a sequence of symmetric positive definite (spd) penalty matrices $\{H_k\}$ instead of the constant penalty β. The convergence of their method was proved under the assumption that the eigenvalues of $\{H_k\}$ are uniformly bounded from below away from zero, and, with finitely many exceptions, the eigenvalues of $H_k - H_{k+1}$ are nonnegative.

In this paper, we continue the Kontogiorgis and Meyer's research [12] and present a modified variable-penalty alternating directions method that allows the eigenvalues of $\{H_k\}$ either to increase or to decrease in each iteration. This can be beneficial in applications. In addition, similarly as in [10], we propose a self-adaptive adjusting rule that leads the method to be more advantageous in practice.

The following notation is used in this paper. We denote by $I_{n\times n}$ the identity matrix in $\mathcal{R}^{n\times n}$. For any real matrix M and vector v, we denote the transposition by M^T and v^T, respectively. The notation $M \succeq 0$ means that M is a positive semi-definite matrix, and $M \succ 0$ means that M is a positive definite matrix. Superscripts such as in v^k refer to specific vectors and are usually iteration indices. The Euclidean norm of vector z will be denoted by $\|z\|$, i.e., $\|z\| = \sqrt{z^T z}$.

2. The General Structure of the Modified Method

Throughout this paper, we call the method by Kontogiorgis and Meyer [12] and our modified method ADM method and MADM method, respectively. To describe the MADM method, we need a non-negative sequence $\{\eta_k\}$ that satisfies $\sum_{k=0}^{\infty} \eta_k < \infty$.