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NUMERICAL DISSIPATION FOR THREE-POINT DIFFERENCE SCHEMES TO HYPERBOLIC EQUATIONS WITH UNEVEN MESHES *1)

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Abstract

The widely used locally adaptive Cartesian grid methods involve a series of abruptly refined interfaces. The numerical dissipation due to these interfaces is studied here for three-point difference approximations of a hyperbolic equation. It will be shown that if the wave moves in the fine-to-coarse direction then the dissipation is positive (stabilizing), and if the wave moves in the coarse-to-fine direction then the dissipation is negative (destabilizing).

Key words: Refined interfaces, Numerical dissipation, Three-point difference approximation, Hyperbolic equation.

1. Introduction

In the adaptive Cartesian grid method [1, 2, 3, 4, 5, 9, 10, 11, 13, 18, 20, 25], the entire grid is composed of divided zones (which will be called subgrids for convenience) each having a uniform mesh size and with abrupt mesh refinement at the interfaces. The adaptive Cartesian grid method shares some common feature with the multilevel methods originally proposed by Brandt and then evolved to the well-known multigrid method for convergence acceleration, see, e.g., [6, 17, 19].

A particular feature of the abrupt refinement method is the existence of multiple refinement interfaces which are separated by subgrids of uniform mesh size. Little attention has been paid to the stability and accuracy of the abrupt method with multiple interfaces, though the single interface problem was addressed long before, see, e.g., [2, 7]. In this paper we will address the question of mesh refinement induced dissipation, which is closely related to the stability of the difference approximation. Some good dissipation analysis can be found in [12, 21, 23]. Normally the influence of the abrupt interfaces is coupled with the treatment of the exterior boundaries. But in this paper we will ignore the influence of the treatment of the exterior boundaries. This simplification will be stated throughout the paper in its suitable form whenever needed.

Here we only consider three-point difference equations with conservative treatment everywhere (inside subgrid and at interface). The difference approximations on both smoothly refined grid and abruptly refined grid, based on the same discretization procedure, are presented in Section 2. Section 3 is devoted to the study of mesh refinement induced dissipation under the framework of semidiscrete scheme. In Section 4 we will consider the fully discrete scheme in order to analyze how the mesh refinement influences the total dissipation. To this end we first we use an energy method to study the necessary condition for energy decreasing (or stability). Then we perform an eigenvalue analysis (including a von Neumann stability analysis for a particular case) in order to study sufficient stability conditions for the Lax-Wendroff scheme.

The conclusions of this paper are summrized in the following two Theorems.

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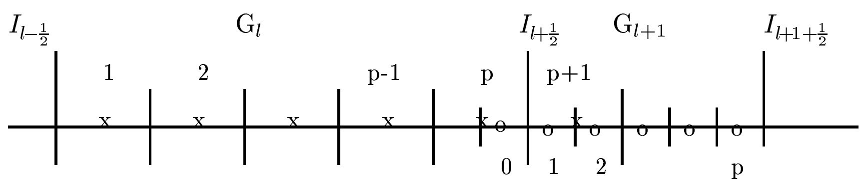

Figure 1: Abrupt refinement grid.

Theorem 1.1. *For a general semi-discrete three-point difference approximation with uneven mesh spacing, if the wave moves in the fine-to-coarse direction then the dissipation is positive (stabilizing) , and if the wave moves in the coarse-to-fine direction then the dissipation is negative (de-stabilizing). Moreover, the amount of dissipation is insenstive to the subgrid width if the total refinement degree is fixed.*

Theorem 1.2. *For the fully discrete Lax-Wendroff scheme, the lower bound of the stability region is increased by mesh refinement, while the upper bound is reduced.*

2. Three-point Difference Approximations on Smooth and Abrupt Refinement Grids

2.1. Smooth Refinement Grid and Abrupt Refinement Grid

On a smoothly refined grid , let h_l be the size of mesh l with $0 \leq l \leq L$. Conventionally, if $r = \frac{h_l}{h_{l-1}}$ is constant independent of l, then the refinement is said to be geometric. In practice we may have $r = \frac{h_l}{h_{l-1}}$ depending on l, but such irregular situations have rarely been investigated theoretically.

On an abruptly refined grid, as displayed in Fig.1, the entire grid is composed of a certain number of subgrids with different mesh sizes. Let h_l be the mesh size on subgrid G_l with $0 \leq l \leq L$. We assume $h_l = h_0 r^l$ with $r < 1$ and that the number of grid points in each subgrid is constant and equal to p, which we will call the subgrid width. For convenience, let us define the total refinement degree by $r_T = h_L/h_0$. For geometrical refinement, the local refinement degree $r = h_{l+1}/h_l$ is related to r_T by

$$r = r_T^{\frac{1}{L}} \tag{2.1}$$

In this paper we will only consider geometrical refinement. Note that the smooth refinement method can be considered as a particular case of the abrupt mesh refinement method with $p = 1$ and $r \to 1$. In deriving the difference equations we also require that the difference equation on an abrupt grid reduces to that on the smooth refinement grid when $p = 1$.

For purpose of studying the numerical dissipation, it is sufficient here to consider the following scalar equation:

$$u_t + au_x = 0 \tag{2.2}$$

approximated by a three-point difference scheme. In order for the results to be useful for nonlinear problems, we require the treatment to be conservative. Also, for the comparison to be meaningful, the difference approximations should be designed from the same discretization methodology for both of the smooth and abrupt refinement methods so that they each reduce to the other when the grid is uniform.