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A V-CYCLE MUTIGRID FOR QUADRILATERAL ROTATED Q_1 ELEMENT WITH NUMERICAL INTEGRATION*[1)]

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Abstract

In this paper, a V-cycle multigrid method is presented for quadrilateral rotated Q_1 elements with numerical integration.

Key words: Multigrid, Rotated Q_1 elements, Numerical integration.

1. Introduction

The rotated Q_1 nonconforming element first proposed and used to solve the Stokes problem by Rannacher and Turek in [12]. Klouček, Li and Luskin have implemented it to simulate the martensitic crystals with microstructures [9], [10]. Recently, Shi and Ming [14] gave a detailed mathematics analysis for this element under the bi-section condition for mesh subdivisions, which was first introduced by Shi [13] for analyzing the quadrilateral Wilson element. Meanwhile they also proposed some effective numerical quadrature schemes for this element[14]. Moreover, they have succeeded in using this element for the Mindlin-Reissner plate problem [11]. Quasi-optimal maximum norm estimations for the quadrilateral rotated Q_1 element approximation of Navier-Stokes equations were established in [17].

In this paper, we will investigate multigrid methods for solving discrete algebraic equations obtained by use of the quadrilateral rotated Q_1 elements. An effective V-cycle multigrid algorithm is presented with numerical integrations. A uniform convergence factor is obtained. A similar idea has been exploited for the Wilson nonconforming element [15] and the TRUNC plate element [16]. We also mention that some nonconforming multigrid algorithms for the second order problem are studied in early papers, see [1], [6] for P_1 nonconforming element, and [8] for the rectangular rotated Q_1 element.

The outline of the paper is as follows. In section 2, we introduce the quadrilateral rotated Q_1 element. In the last section an effective V-cycle multigrid algorithm is presented.

2. Quadrilateral Rotated Q_1 Elements

We consider the following general 2-order elliptic boundary value problem over a convex polygonal domain in R^2:

$$\begin{aligned} \pounds u = -(\partial_x(a_{11}\partial_x u) + \partial_y(a_{12}\partial_x u) + \partial_x(a_{12}\partial_y u) + \partial_y(a_{22}\partial_y u)) + au &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where the coefficients $a_{11}, a_{12}, a_{22},\ a \in W^{1,\infty}(\Omega)$, and $a \geq 0$, the right hand term $f \in W^{1,q}(\Omega), q \geq 2$, $W^{1,\infty}(\Omega)$ and $W^{1,q}(\Omega)$ are the usual Sobolev spaces.

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We assume that the differential operator $\mathcal{L}$ is uniformly elliptic, i.e. there exists a positive constant c such that

$$c^{-1}(\xi_1^2+\xi_2^2) \le \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \le c(\xi_1^2+\xi_2^2)$$

for all points $(x,y) \in \bar{\Omega}$ and real vectors (ξ_1, ξ_2).

The weak form of this problem is to find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \tag{2.1}$$

where

$$a(u,v) = \int_\Omega [a_{11}\partial_x u \partial_x v + a_{12}(\partial_x u \partial_y v + \partial_y u \partial_x v) + a_{22}\partial_y u \partial_y v + auv]dxdy.$$

Let Γ_h be a partition of the convex polygonal $\overline{\Omega}$ by convex quadrilaterals. Denote $\Gamma = \partial\Omega$. We define by P_k the space of polynomials of degrees no more than k, and by Q_k the space of polynomials of degrees no more than k in each variable. Let the diameter of K be h_K and assume that $h_K \le h$. As in Figure 1, we denote the four vertices of K by $P_i(x_i, y_i), 1 \le i \le 4$, and the sub-triangle of K with vertices P_{i-1}, P_i, and P_{i+1} by T_i (the index of P_i is modulo 4). Define $\rho_K = \max_{1\le i\le 4}$ (diameter of the circles inscribed in T_i). It is assumed that the partition satisfies the assumption: there exists a constant $\sigma > 2$ independent of h such that

$$h_K \le \sigma\rho_K. \tag{2.2}$$

Note that this assumption is equivalent to the usual regularity condition for quadrilateral partitions (see [7], pp. 247). Let $\hat{K} = [-1,1]\times[-1,1]$ be the reference square having the vertices $\hat{P}_i (1 \le i \le 4)$, then there exists a unique mapping $F_K \in Q_1(\hat{K})$ given by

$$x^K = \sum_{i=1}^{4} x_i N_i(\xi,\eta), \quad y^K = \sum_{i=1}^{4} y_i N_i(\xi,\eta),$$

where

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta), \quad N_3 = \frac{1}{4}(1+\xi)(1+\eta), \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

such that $F_K(\hat{p}_i) = p_i, 1 \le i \le 4, \quad F_K(\hat{K}) = K$. We also denote $e_1 = \overline{P_4P_1}, e_2 = \overline{P_1P_2}, e_3 = \overline{P_2P_3}, e_4 = \overline{P_3P_4}$.

To each function $\hat{v}(\xi,\eta)$ defined on $\hat{K}$, we associate a function v on K such that $\hat{v} = v \circ F_K$.

In the following, we list some geometric properties of an arbitrary quadrilateral mesh:

$$\begin{array}{ll}
x^K = a_0 + a_1\xi + a_2\eta + a_{12}\xi\eta, & y^K = b_0 + b_1\xi + b_2\eta + b_{12}\xi\eta. \\
4a_0 = x_1 + x_2 + x_3 + x_4, & 4b_0 = y_1 + y_2 + y_3 + y_4. \\
4a_1 = -x_1 + x_2 + x_3 - x_4, & 4b_1 = -y_1 + y_2 + y_3 - y_4. \\
4a_2 = -x_1 - x_2 + x_3 + x_4, & 4b_2 = -y_1 - y_2 + y_3 + y_4. \\
4a_{12} = x_1 - x_2 + x_3 - x_4, & 4b_{12} = y_1 - y_2 + y_3 - y_4.
\end{array}$$

$$DF_K(\xi,\eta) = \begin{pmatrix} a_1 + a_{12}\eta & a_2 + a_{12}\xi \\ b_1 + b_{12}\eta & b_2 + b_{12}\xi \end{pmatrix}$$

and the Jacobi of F_K is $J_K(\xi,\eta) = det(DF_K) = J_0^K + J_1^K\xi + J_2^K\eta$, where, $J_0^K = a_1b_2 - a_2b_1, J_1^K = a_1b_{12} - a_{12}b_1, J_2^K = a_{12}b_1 - a_2b_{12}$. Denote the inverse of F_K by F_K^{-1}, then

$$(DF_K)^{-1}(\xi,\eta) = \frac{1}{J_K(\xi,\eta)} \begin{pmatrix} b_2 + b_{12}\xi & -a_2 - a_{12}\xi \\ -b_1 - b_{12}\eta & a_1 + a_{12}\eta \end{pmatrix}$$