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THE COUPLING OF NATURAL BOUNDARY ELEMENT AND FINITE ELEMENT METHOD FOR 2D HYPERBOLIC EQUATIONS *1)

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Abstract

In this paper, we investigate the coupling of natural boundary element and finite element methods of exterior initial boundary value problems for hyperbolic equations. The governing equation is first discretized in time, leading to a time-step scheme, where an exterior elliptic problem has to be solved in each time step. Second, a circular artificial boundary Γ_R consisting of a circle of radius R is introduced, the original problem in an unbounded domain is transformed into the nonlocal boundary value problem in a bounded subdomain. And the natural integral equation and the Poisson integral formula are obtained in the infinite domain Ω_2 outside circle of radius R. The coupled variational formulation is given. Only the function itself, not its normal derivative at artificial boundary Γ_R, appears in the variational equation, so that the unknown numbers are reduced and the boundary element stiffness matrix has a few different elements. Such a coupled method is superior to the one based on direct boundary element method. This paper discusses finite element discretization for variational problem and its corresponding numerical technique, and the convergence for the numerical solutions. Finally, the numerical example is presented to illustrate feasibility and efficiency of this method.

Key words: Hyperbolic equation, Natural boundary reduction, Finite element, Coupling, Exterior problem.

1. Introduction

In many fields of scientific and engineering computing, problems in unbounded spatial domains are encountered frequently, such as acoustic waves, electromagnetics wave guides, aerodynamics, and meteorology, and so on. Such problems pose a unique challenge to computation, since their domains are unbounded. Although we can apply classical boundary element methods (BEM) or boundary integral methods (BIM) to solve these problems in unbounded domains, in practice a great many singular integrations usually need to be calculated. At the same time, we make the integrations about time interval for the time-independent problems while the problems are discretized in time. Therefore, it takes a lot of time to deal with the original problem. The natural boundary element method initiated and developed by K Feng and D Yu (see $[1-6]$) has some distinctive advantages comparing with classical boundary element methods. One of them is fully compatible with finite element method, and it can be coupled with finite element method naturally and directly. The coupling of natural boundary element and finite element method for the elliptic problems, we can refer to [6,7,8]. Mathematical theory

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of natural boundary element method for the elliptic problems is being perfected (see [8]). At present, authors have made some developments in natural boundary element method for the parabolic problems(see [9]), and been investigating into the problems related to this.

In this paper, the coupling of natural boundary element and finite element method for hyperbolic exterior initial boundary value problems in R^2. A circular artificial common boundary Γ_R, which consists of a circle of radius R large enough, is introduced. It divides the domain into two subregions, a bounded inner region Ω_1 (bounded annular region by Γ_0 and Γ_R) and a regular unbounded region Ω_2 (unbounded domain outside circle Γ_R). We obtain the natural integral equation on boundary Γ_R and corresponding Poisson integral formula of the subproblem over unbounded domain Ω_2 by the natural boundary reduction. Only the function itself, not its normal derivative at the common boundary Γ_R, appears in the variational equation obtained. Not only the boundary element stiffness matrix is symmetric, but also its elements have explicit forms. We only calculate a few different elements, and can get the boundary element stiffness matrix. It is easy to be implemented on calculation and storage.

Following is the outline of this paper. In section 2 we first state the problem under consideration . Second we discretize the problem in time, leading to a time-stepping scheme, where an exterior elliptic problem has to be solved in each time step. In section 3 we introduce a circular common boundary Γ_R, and obtain natural integral equation in the unbounded domain outside circle Γ_R. The problem in an unbounded domain is transformed into the nonlocal boundary value problem in a bounded domain. We then introduce the corresponding variational formulation and show that the variational problem has a unique solution. We also give some properties of the natural integral operator $\mathcal{K}_\lambda^c$, and the bilinear form $\hat{D}_2(\cdot,\cdot)$ obtained by operator $\mathcal{K}_\lambda^c$. The finite element discretization is employed to solve the variational problem in section 4. The convergence for the numerical solution is taken into account. In section 5 we present some numerical results to illustrate feasibility and efficiency of our method.

2. Statement of the Problem and Time Discretization

Let Γ_0 be a closed curve in plane, and Ω be an exterior domain with Γ_0 as its boundary. For any fixed positive number T, writing $J := (0,T]$. Consider the following initial boundary problems:

$$\begin{cases} u_{tt} - a^2\Delta u = f(x,t), & (x,t) \in \Omega \times J; \\ \dfrac{\partial u(x,t)}{\partial n} = g(x,t), & (x,t) \in \Gamma_0 \times J; \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), & x \in \Omega. \end{cases} \tag{2.1}$$

where $u(x,t)$ is the unknown function, u_t and u_{tt} denote the first derivative and the second derivative with respect to time t, respectively. a is a given positive constant (It is usually wave speed). $f(x,t)$, $g(x,t)$, $\varphi(x)$ and $\psi(x)$ are all given functions. $\frac{\partial}{\partial n}$ is the normal derivative operator on Γ_0 (Here n is the outer unit normal vector on boundary Γ_0 of domain Ω toward the interior domain with Γ_0 as its boundary). Moreover, we assume the function $u(x,t)$ is bounded at infinity. However, there is no need in a "radiation condition" at infinity to complete the statement of the problem (see [10]).

Let τ be the time-step, and write $t_k = k\cdot\tau$, $u^k(x) = u(x,t_k)$, $z^k(x) = u_t(x,t_k)$, $w^k(x) = u_{tt}(x,t_k)$.

$$\lambda := (\tau a\sqrt{\alpha})^{-1}, \qquad \tilde{u}^{k+1} := u^k + \tau\cdot z^k + \frac{\tau^2}{2}(1-2\alpha)w^k,$$
$$\tilde{z}^{k+1} := z^k + (1-\beta)\tau w^k, \qquad \tilde{f}^{k+1} := -\tilde{u}^{k+1} - \alpha\tau^2 f^{k+1}$$

Here $\alpha \in (0,\frac{1}{2}]$ and $\beta \in [0,1]$, $\quad k = 0,1,2,\cdots,[T/\tau]-1$. Then the original problem (2.1) can be reduced to the following one (discrete problem in time t):