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ASYMPTOTICALLY OPTIMAL SUCCESSIVE OVERRELAXATION METHODS FOR SYSTEMS OF LINEAR EQUATIONS*[1)]

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Abstract

We present a class of asymptotically optimal successive overrelaxation methods for solving the large sparse system of linear equations. Numerical computations show that these new methods are more efficient and robust than the classical successive overrelaxation method.

Key words: Successive Overrelaxation Methods, System of Linear Equations.

1. Introduction

Consider the solution of system of linear equations

$$Ax = b, \quad A \in \mathbb{R}^{n\times n} \text{ nonsingular, and } x, b \in \mathbb{R}^n, \tag{1}$$

where the coefficient matrix $A \in \mathbb{R}^{n\times n}$ is large sparse, and usually, has certain particular structures and properties, $b \in \mathbb{R}^n$ is a given right-hand-side vector, and $x \in \mathbb{R}^n$ is the unknown vector.

The successive overrelaxation (SOR) method [9] provides one powerful tool for solving the system of linear equations (1), in particular, when an optimal, or at least, a nearly optimal relaxation factor is easily obtainable. However, except we have an analytic formula about the optimal relaxation factor for the consistently ordered p-cyclic matrix [9, 6, 2], we know little about its choice in actual computations for a general matrix. Even the analytic formula is practically unapplicable, because it involves the spectral radius of the corresponding Jacobi iteration matrix, whose computation is considerably costly and complicated. This heavily restricts efficient applications of the SOR method to a wider range of real-world problems.

In this paper, by choosing the relaxation factor in a dynamic fashion according to known information at the current iterate step, we propose a class of new SOR methods, called as asymptotically optimal SOR methods (AOSOR methods), for solving the system of linear equations (1).

The AOSOR methods determine the relaxation factors iteratively through minimizing either the A-norm of the error when the coefficient matrix $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, or the 2-norm of the residual when it is a general unsymmetric nonsingular matrix, at each step of their iterates, with a reasonably extra cost. In actual computations, they show better numerical behaviours than the SOR method for both symmetric positive definite matrix and general unsymmetric nonsingular matrix. Numerical experiments show that the

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new AOSOR methods are feasible, efficient and robust for solving large sparse system of linear equations (1).

2. The SOR Method and its Properties

Without loss of generality, we assume that the diagonal matrix of the matrix $A \in \mathbb{R}^{n\times n}$ is the identity I. Let $-L$ and $-U$ be strictly lower and strictly upper triangular matrices of the matrix $A \in \mathbb{R}^{n\times n}$, respectively. Then it holds that $A = I - L - U$. The SOR method for solving the system of linear equations (1) can be expressed as

$$x^{p+1} = \mathcal{L}(\omega)x^p + g(\omega),$$

where

$$\mathcal{L}(\omega) = (I - \omega L)^{-1}((1-\omega)I + \omega U), \quad g(\omega) = \omega(I - \omega L)^{-1}b. \tag{2}$$

If we further introduce matrices

$$\mathcal{M}(\omega) = \frac{1}{\omega}(I - \omega L), \quad \mathcal{N}(\omega) = \frac{1}{\omega}((1-\omega)I + \omega U), \tag{3}$$

then it holds that

$$\mathcal{L}(\omega) = \mathcal{M}(\omega)^{-1}\mathcal{N}(\omega), \quad g(\omega) = \mathcal{M}(\omega)^{-1}b.$$

If $\omega = 1$, the SOR method simplifies to the Gauss-Seidel method. And various generalizations of the SOR method can be found in [1, 3, 4, 7, 8].

It is well-known that the SOR method converges to the unique solution x^* of the system of linear equations (1) when the coefficient matrix $A \in \mathbb{R}^{n\times n}$ is an M-matrix, an H-matrix, an irreducibly diagonally dominant matrix, and a symmetric positive definite matrix, respectively, under certain restrictions on the relaxation factor. More precisely, we have the following conclusions.

Theorem 2.1. *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix, and its diagonal entries be all nonzero. Denote $D = \mathrm{diag}(A)$, $B = D - A$, and $J = D^{-1}B$. Then the SOR method is convergent to the unique solution of the system of linear equations (1), if*

(a) $A \in \mathbb{R}^{n\times n}$ is an M-matrix, and $0 < \omega < \frac{2}{1+\rho(J)}$; [5]

(b) $A \in \mathbb{R}^{n\times n}$ is an H-matrix, and $0 < \omega < \frac{2}{1+\rho(|J|)}$; [5]

(c) $A \in \mathbb{R}^{n\times n}$ is an irreducibly diagonally dominant matrix, and $0 < \omega < \frac{2}{1+\rho(|J|)}$; [5]

(d) $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, and $0 < \omega < 2$. [9]

Here, $\rho(\cdot)$ and $|\cdot|$ denote the spectral radius and the absolute value of the corresponding matrix, respectively.

Moreover, for the consistently ordered p-cyclic matrix class, we have the following precise description about the optimum relaxation factor of the SOR method.

Theorem 2.2[9]. *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular and consistently ordered p-cyclic matrix, with nonzero diagonal entries. Denote $D = \mathrm{diag}(A)$, $B = D - A$, and $J = D^{-1}B$. If $\omega \neq 0$, and λ is a nonzero eigenvalue of the matrix $\mathcal{L}(\omega)$ of (2) and if μ satisfies*

$$(\lambda + \omega - 1)^p = \lambda^{p-1}\omega^p\mu^p, \tag{4}$$

then μ is an eigenvalue of the Jacobi iteration matrix J. Conversely, if μ is an eigenvalue of J and λ satisfies (4), then λ is an eigenvalue of $\mathcal{L}(\omega)$.

Moreover, the optimum relaxation factor ω_{opt} which minimizes the asymptotic convergence rate of the SOR method is the unique positive real root (less than $p/(p-1)$) of the equation

$$(\rho(J)\omega_{opt})^p = (p^p(p-1)^{1-p})(\omega_{opt} - 1), \tag{5}$$