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LONG TIME ASYMPTOTIC BEHAVIOR OF SOLUTION OF IMPLICIT DIFFERENCE SCHEME FOR A SEMI-LINEAR PARABOLIC EQUATION *1)

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Abstract

In this paper, the solution of back-Euler implicit difference scheme for a semi-linear parabolic equation is proved to converge to the solution of difference scheme for the corresponding semi-linear elliptic equation as t tends to infinity. The long asymptotic behavior of its discrete solution is obtained which is analogous to that of its continuous solution. At last, a few results are also presented for Crank-Nicolson scheme.

Key words: Asymptotic behavior, Implicit difference scheme, Semi-linear parabolic equation, Convergence.

1. Introduction

Consider the following initial-boundary value problem

$$\frac{\partial u}{\partial t} = \triangle u - \phi(u) + f(x,y), \quad (x,y,t) \in \Omega \times R_+, \tag{1.1.1}$$

$$u|_{\partial\Omega} = 0, \tag{1.1.2}$$

$$u|_{t=0} = u_0(x,y), \quad (x,y) \in \Omega, \tag{1.1.3}$$

where $\triangle$ is Laplac's operator, Ω is a rectangular $[0,l]^2$, $R_+ = (0,\infty)$, $\phi'(u) \geq 0$. As t tends to ∞ and $\phi'(u)$ satisfies some conditions, the solution of (1.1) converges to that of the following semi-linear elliptic boundary value problem

$$\triangle\, u - \phi(u) + f(x,y) = 0, \quad (x,y) \in \Omega, \tag{1.2.1}$$

$$u|_{\partial\Omega} = 0, \tag{1.2.2}$$

Comparing to the case of continuous problem, it is very interesting to discuss the asymptotic behavior of discrete solution of difference scheme for (1.1). For one-dimensional problem (1.1) and $\phi(u) = u^3$, Hui Feng and Long-jun Shen proved the solution of backward Euler difference scheme and forward Euler difference scheme converge to the solution of the difference scheme for the relevant nonlinear stationary problem as t tends to infinity and obtained the long time asymptotic behavior of discrete solution in [1] and [2] respectively by energy method.

In this paper, we consider back-Euler implicit difference scheme for (1.1) and give the asymptotic error estimates by using Browder fixed point theorem, maximum principle and energy method. For the Crank-Nicolson difference scheme, some similar results are also given.

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Let $h, \Delta t_n$ be the space step-size and the time step-size respectively, $h = l/J$, where J is an integer. Denote $\Omega_h = \{(x_i, y_j) \mid x_i = ih, y_j = jh, 0 \le i, j \le J\}$ and $H = \{w \mid w = \{w_{ij}\}_{i,j=0}^{J},\ w_{i0} = w_{iJ} = w_{0j} = w_{Jj} = 0, 0 \le i, j \le J\}$.

For $w \in H$, introduce the following notations:

$$\delta_x w_{i+\frac{1}{2},j} = (w_{i+1,j} - w_{ij})/h, \qquad \delta_y w_{i,j+\frac{1}{2}} = (w_{i,j+1} - w_{ij})/h,$$
$$\delta_x^2 w_{ij} = (w_{i+1,j} - 2w_{ij} + w_{i-1,j})/h^2, \qquad \delta_y^2 w_{ij} = (w_{i,j+1} - 2w_{ij} + w_{i,j-1})/h^2,$$
$$\triangle_h w_{ij} = \delta_x^2 w_{ij} + \delta_y^2 w_{ij},$$
$$\|w\|_C = \max_{1 \le i,j \le J-1} |w_{ij}|, \qquad \|w\| = \sqrt{h^2 \sum_{i=1}^{J-1} \sum_{j=1}^{J-1} (w_{ij})^2},$$
$$\|\delta_h w\| = \sqrt{h^2 \left[\sum_{i=0}^{J-1} \sum_{j=1}^{J-1} (\delta_x w_{i+\frac{1}{2},j})^2 + \sum_{i=1}^{J-1} \sum_{j=0}^{J-1} (\delta_y w_{i,j+\frac{1}{2}})^2 \right]},$$
$$\|\Delta_h w\| = \sqrt{h^2 \sum_{i=1}^{J-1} \sum_{j=1}^{J-1} (\Delta_h w_{ij})^2}.$$

It is easy to know that $\|w\|_C$, $\|w\|$, $\|\delta_h w\|$ and $\| \triangle_h w\|$ are all norms of the space H. In addition, if $v \in H$ and $w \in H$, we define the inner product

$$(v, w) = h^2 \sum_{i=1}^{J-1} \sum_{j=1}^{J-1} v_{ij} w_{ij}.$$

It is obvious that

$$\|w\| = \sqrt{(w, w)}.$$

The back-Euler implicit difference scheme for (1.1) we will consider is as follows

$$\frac{u_{ij}^n - u_{ij}^{n-1}}{\Delta t_n} = \triangle_h u_{ij}^n - \phi(u_{ij}^n) + f(x_i, y_j),$$
$$1 \le i, j \le J-1, n = 1, 2, 3, \cdots \tag{1.3.1}$$
$$u_{i0}^n = u_{iJ}^n = u_{0j}^n = u_{Jj}^n = 0, 0 \le i, j \le J, n = 1, 2, 3, \cdots, \tag{1.3.2}$$
$$u_{ij}^0 = u_0(x_i, y_j),\ 0 \le i, j \le J. \tag{1.3.3}$$

For (1.2), we construct the following difference scheme

$$\triangle_h u_{ij}^* - \phi(u_{ij}^*) + f(x_i, y_j) = 0,\ 1 \le i, j \le J-1, \tag{1.4.1}$$
$$u_{i0}^* = u_{iJ}^* = u_{0j}^* = u_{Jj}^* = 0, 0 \le i, j \le J, \tag{1.4.2}$$

In next section, the difference schemes (1.3) and (1.4) are proved to have unique and bounded discrete solutions respectively. In section 3, the asymptotic error estimates are given, from which it is known that the solution of (1.3) converges to the solution of (1.4).

The main result of this paper is Theorem 3.1 proved in section 3.

2. Preliminary Results

For our need, we list following lemmas.