Journal of Computational Mathematics, Vol.21, No.5, 2003, 681–688.

CURVATURE COMPUTATIONS OF 2-MANIFOLDS IN $I\!R^k$ *1)

Guo-liang Xu
(*Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Chandrajit L. Bajaj
(*Department of Computer Science, University of Texas, Austin, TX 78712*)

Abstract

In this paper, we provide simple and explicit formulas for computing Riemannian curvatures, mean curvature vectors, principal curvatures and principal directions for a 2-dimensional Riemannian manifold embedded in $I\!R^k$ with $k \geq 3$.

Key words: Riemannian curvature, Mean curvature vector, Principal curvatures, Principal directions.

1. Introduction

The concepts of Riemannian curvatures, mean curvature vectors and principal curvatures have been developed in the field of *Riemannian Geometry*. These concepts are respectively generalizations of Gaussian curvatures, mean curvatures and principal curvatures for the classical surfaces in $I\!R^3$. It is well known that these three types curvatures for classical surfaces are extremely important notions in *Computational Geometry*, *Computer Graphics*, *Image Processing* and *Computer Added Geometric Design*. Their counterparts for 2-dimensional Riemannian manifold (abbreviated as 2-manifold) embedded in $I\!R^k$ are, as expected, equally important. Indeed, we have found that these concepts play an important role in the fields of image processing ([5, 7, 1]) and function diffusion ([2, 3]). However, the general frame of Riemannian geometry makes these curvatures difficult to calculate.

We provide simple and explicit formulas for computing Riemannian curvatures, mean curvature vectors, principal curvatures and principal directions for a 2-manifold embedded in $I\!R^k$ with $k \geq 3$. These formulas are simple compared to those found in Riemannian geometry literature ([4, 6, 8]). Individuals with little knowledge of Riemannian geometry, but who are familiar with vector computations in the Euclidean space, can easily understand and use them. Even though the starting point of the derivation of these formulas involves the use of Riemannian geometry, we have tried to minimize its use while keeping the derivation as precise as possible.

It may seem trivial for people working in the field of Riemannian geometry to derive these curvature formulas for the 2-manifold, however we have not seen these formulas presented in a simple and precise enough manner to fulfill our needs.

2. Curvature Formulas

The aim of this section is to provide readers with a quick reference for the curvature computation formulas. The detail derivation of these formulas are given in the section that follows.

* Received July 31, 2001.

1) The first author was supported in part by NSF (10241004) of China and National Innovation Fund 1770900, Chinese Academy of Sciences; the second author was supported in part by NSF grants CCR 9732306 and KDI-DMS-9873326.

Let M be a 2-dimensional Riemannian manifold in $\mathbb{R}^k$ with a Riemannian metric defined by the scalar inner product. Let (ξ_1, ξ_2) be a local coordinate system of the 2-manifold M at the point $x \in M$. Then $x \in \mathbb{R}^k$ can be expressed as

$$x = [x_1(\xi_1, \xi_2), \cdots, x_k(\xi_1, \xi_2)]^T. \tag{2.1}$$

Let $t_i = \frac{\partial x}{\partial \xi_i}$, $t_{ij} = \frac{\partial^2 x}{\partial \xi_i \partial \xi_j}$, $g_{ij} = t_i^T t_j$, and

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}, \quad Q = I - [t_1, t_2] G^{-1} [t_1, t_2]^T \in \mathbb{R}^{k \times k},$$

where

$$G^{-1} = \frac{1}{\det(G)} \begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{bmatrix}.$$

Then we have the following formulas:

Riemannian Curvature:

$$K(x) = \frac{t_{11}^T Q t_{22} - t_{12}^T Q t_{12}}{\det(G)}. \tag{2.2}$$

The Riemannian curvature is a counterpart of the Gaussian curvature of the classical surface. If $k = 3$, the Riemannian curvature coincides with the Gaussian curvature for surfaces.

Mean Curvature Vector:

$$H(x) = \frac{Q(g_{22} t_{11} + g_{11} t_{22} - 2 g_{12} t_{12})}{2 \det(G)}. \tag{2.3}$$

The mean curvature vector is a vector in the normal space. If $k = 3$, the mean curvature vector is in the normal direction, and its length is the classical mean curvature of the surface.

Principal Curvatures and Principal directions:

To obtain formulas for the principal curvatures and the principal directions, we first introduce an auxiliary result: *Let $A = (a_{ij})_{i,j=1}^2 \in \mathbb{R}^{2 \times 2}$ be a symmetric matrix. Then the eigenvalues of A are*

$$\lambda_{\pm} = \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 + 4 a_{12}^2}}{2} \tag{2.4}$$

and the corresponding eigenvectors are $[cos\theta_{\pm}, sin\theta_{\pm}]^T$, where $\theta_{\pm}$ are given (modulo π) by

$$\theta_+ = \frac{1}{2} \arctan \frac{2 a_{12}}{a_{11} - a_{22}}, \quad \theta_- = \theta_+ + \frac{\pi}{2}. \tag{2.5}$$

Now we give formulas for computing the principal curvatures and the principal directions. Let $h(x) = H(x)/\|H(x)\|$,

$$A = \Lambda^{-\frac{1}{2}} K F_h K^T \Lambda^{-\frac{1}{2}} \in \mathbb{R}^{2 \times 2}, \quad [u_1, u_2] = [t_1, t_2] K^T \Lambda^{-\frac{1}{2}}, \tag{2.6}$$

where $F_h = -\left(t_{ij}^T\ h(x)\right)_{ij=1}^2$, $K \in \mathbb{R}^{2 \times 2}$ and $\Lambda \in \mathbb{R}^{2 \times 2}$ are defined by

$$G = K^T \Lambda K, \quad K^T K = I, \quad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2) \tag{2.7}$$

and they can be computed by (2.4)–(2.5). Let A be expressed, by virtue of (2.4) and (2.5), as

$$A = P\ \mathrm{diag}(k_1, k_2)\ P^T, \quad \text{with} \quad P^T P = I. \tag{2.8}$$

Then k_1 and k_2 are the principal curvatures and v_1 and v_2, defined by

$$[v_1, v_2] := [u_1, u_2] P = [t_1, t_2] K^T \Lambda^{-\frac{1}{2}} P, \tag{2.9}$$

are the corresponding principal directions with respect to the direction vector h.

Again, the principal curvatures and the principal directions are the counterparts of the same concepts for surfaces. If $k = 3$, they are the same.