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SUPERCONVERGENCE OF LEAST-SQUARES MIXED FINITE ELEMENT FOR SECOND-ORDER ELLIPTIC PROBLEMS *1)

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Abstract

In this paper the least-squares mixed finite element is considered for solving second-order elliptic problems in two dimensional domains. The primary solution u and the flux $\boldsymbol{\sigma}$ are approximated using finite element spaces consisting of piecewise polynomials of degree k and r respectively. Based on interpolation operators and an auxiliary projection, superconvergent H^1-error estimates of both the primary solution approximation u_h and the flux approximation $\boldsymbol{\sigma}_h$ are obtained under the standard quasi-uniform assumption on finite element partition. The superconvergence indicates an accuracy of $O(h^{r+2})$ for the least-squares mixed finite element approximation if Raviart-Thomas or Brezzi-Douglas-Fortin-Marini elements of order r are employed with optimal error estimate of $O(h^{r+1})$.

Key words: Elliptic problem, Superconvergence, Interpolation projection, Least-squares mixed finite element.

1. Introduction

We are concerned with approximate solutions for the representative second-order elliptic boundary-value problem:

$$-\mathrm{div}(A\mathrm{grad}u) + cu = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \Gamma, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ is a open bounded domain with boundary Γ and A is a positive definite matrix of coefficients. Introducing the flux $\boldsymbol{\sigma} = -A\mathrm{grad}u$, the problem may be recast as the first order system

$$\boldsymbol{\sigma} + A\mathrm{grad}u = 0 \quad \text{in } \Omega, \tag{1.3}$$

$$\mathrm{div}\boldsymbol{\sigma} + cu = f \quad \text{in } \Omega, \tag{1.4}$$

$$u = 0 \quad \text{on } \Gamma. \tag{1.5}$$

In many applications such as reservoir simulation, second-order elliptic equations are coupled with other partial differential equations through the velocity terms. So, The mixed finite element methods are usually used. The classical mixed method for (1.3)-(1.5) is based on the stationary principle for a saddle-point problem and is subject to the inf-sup condition on the spaces for u and $\boldsymbol{\sigma}$ (see Brezzi [1]), This implies certain restrictions on the polynomial degree k and r for the element bases defining approximations u_h and $\boldsymbol{\sigma}_h$ respectively. In the least-squares mixed (LSM) approach a least-squares residual minimization is introduced for the mixed system

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(1.3)-(1.5) of u and $\boldsymbol{\sigma}$. The finite element approximation yields a symmetric discrete system for the solution $u_h \in V_h$ and $\boldsymbol{\sigma} \in \boldsymbol{W}_h$, where V_h and $\boldsymbol{W}_h$ are the respective approximation subspaces which needn't to be subject to the consistency requirement. In [16-18], Pehlivanov et al. presented a least-squares mixed (LSM) finite elements method for second-order elliptic problems. It have been proved that the LSM method is not subject to the LBB condition and error estimates for various choices of approximation spaces have been obtained.

The objective of this paper is to investigate superconvergence phenomena for second-order elliptic problems by using the LSM method. Such a study is important in applications to mathematical modeling of fluid flow in porous media since the modeling process requires the determination of a very accurate fluid velocity. Various superconvergence results have been established for the mixed finite element for elliptic problems [11-12, 14] and, for miscible displacement problems [2-6, 9, 13]. In the 1990s, Lin et al.[14-15] introduced a so-called interpolation postprocessing technique into the finite element mthods and obtained the globally high-accuracy approximation for solution problems. C.M.Chen and Y.Q.Huang [6] presented an element analysis methods for the high-accuracy theory of the finite element methods.

The paper is organized as follows: In Section 2 we formulate the problem and its LSM finite element approximation and the coerciveness of the bilinear form in appropriate spaces are stated. In Section 3 the interpolation operators and an auxiliary projection are defined and some identity technique results are presented. The superconvergent approximation properties are derived for the LSM method.

2. Problem Formulation and the LSM Approach

We assume that the matrix of coefficients $A = (a_{ij}(x))_{i,j=1}^2$, $x \in \overline{\Omega}$, is positive definite and the coefficients $a_{ij}(x)$ are bounded; i.e. there exist constants α_1 and α_2 such that

$$\alpha_1 \zeta^T \zeta \le \zeta^T A \zeta \le \alpha_2 \zeta^T \zeta, \tag{2.1}$$

for all vectors $\zeta \in \mathbb{R}^2$ and all $x \in \overline{\Omega}$.

The standard notations for Sobolev spaces $H^m(\Omega)$ with norm $||\cdot||_{m,\Omega}$ and seminorms $|\cdot|_{i,\Omega}$, $0 \le i \le m$, are employed throughout. as usual, $L^2(\Omega) = H^0(\Omega)$ and let $(H^m(\Omega))^2$ be the corresponding product space. Also, we shall use the spaces $H^s(\Gamma)$. Let

$$V = \{v \in H^1(\Omega): \quad v = 0 \ \text{ on } \ \Gamma\}.$$

By the Poincaré-Friedrichs inequality

$$||v||_{0,\Omega} \le C_F |v|_{1,\Omega} \quad \text{for all} \quad v \in V. \tag{2.2}$$

Let

$$c_0 = \min\left\{\inf_{x\in\Omega} c(x), \ 0\right\}. \tag{2.3}$$

We make the following assumptions with respect to the coefficients of our equation: there exist constants α_0 and c_1 such that

$$|c(x)| \le c_1 \quad \text{for all} \quad x \in \overline{\Omega}, \tag{2.4}$$

$$0 < \alpha_0 \le \alpha_1 + c_0 C_F^2, \tag{2.5}$$

where C_F is the constant from the Poincaré-Friedrichs inequality above. Hence, the coefficient c(x) may be negative provided that α_1 is sufficiently large.

Let $\boldsymbol{\tau} = (\tau_1, \tau_2)$ be a smooth vector function and $v \in H^1(\Omega)$, we denote that

$$\text{div}\boldsymbol{\tau} = \partial_1 \tau_1 + \partial_2 \tau_2, \qquad \text{grad}v = (\partial_1 v, \partial_2 v).$$

Introducing the following spaces:

$$\boldsymbol{W} = \{\boldsymbol{\tau} \in (L^2(\Omega))^2, \ \ \text{div}\boldsymbol{\tau} \in L^2(\Omega)\}, \tag{2.6}$$