Journal of Computational Mathematics, Vol.22, No.1, 2004, 123–136.

CASCADIC MULTIGRID METHOD FOR ISOPARAMETRIC FINITE ELEMENT WITH NUMERICAL INTEGRATION *1)

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Abstract

The purpose of this paper is to study the cascadic multigrid method for the second-order elliptic problems with curved boundary in two-dimension which are discretized by the isoparametric finite element method with numerical integration. We show that the CCG method is accurate with optimal complexity and traditional multigrid smoother (like symmetric Gauss-Seidel, SSOR or damped Jacobi iteration) is accurate with suboptimal complexity.

Key words: Cascadic multigrid method, Isoparametric element, Numerical integration.

1. Introduction

In this paper, we consider the second-order linear elliptic problems posed over a bounded domain $\Omega \subset R^2$ with curved boundary Γ. The problem can be described as

$$Lu = -\mathrm{div}(a(x)\mathrm{grad}u) + b(x)u \quad = \quad f, \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u \quad = \quad 0, \quad \text{on} \quad \Gamma, \tag{1.2}$$

where $a(x)$ is a (sufficiently smooth) uniformly positive definite matrix in Ω, $b(x)$ is sufficiently smooth and $0 < b \le b(x)$.

The weak form of the problem (1.1)–(1.2) is: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v), \qquad \forall v \in H_0^1(\Omega), \tag{1.3}$$

where

$$a(u,v) = \int_\Omega ((a(x)\nabla u)\cdot\nabla v + b(x)uv)\mathrm{d}x,$$

$$(f,v) = \int_\Omega fv\mathrm{d}x.$$

In this paper, we will need to assume the H^2-regularity on Problem (1.1)–(1.2). We formalize it into assumption (A.1).

A.1 For any $f \in L^2(\Omega)$, the corresponding solution u of Problem (1.1)-(1.2) is in the space $H^2(\Omega) \cap H_0^1(\Omega)$ and there exists a constant C independent of u and f such that

$$\|u\|_{2,\Omega} \le C\|f\|_{0,\Omega}, \qquad \forall f \in L^2(\Omega). \tag{1.4}$$

For the second-order selfadjoint elliptic boundary value problems, taking into account numerical integration, Ciarlet [5], Ciarlet and Raviart [6] and Li [8] obtained the error estimates

* Received October 23, 2001.
1) This research was supported by the National Natural Science Foundation of China under grant 10071015.

in $H^1(\Omega_h)$-norm and $H^1(\Omega)$-norm respectively. In general, Ω_h does not contain Ω and vice versa. Because the identical quadrature scheme is used in [5] [6] and [8], the finite element solution obtained in [8] is the same as that in [5] and [6] respectively. In [8], Li avoid to extend the partial differential equation and only need the H^2-regularity assumption on the differential equation (1.1)-(1.2) to obtain the error estimate in $H^1(\Omega)-$norm. While in [5] and [6], in order to make (4.4.68) in [5] hold and obtain the error estimate in $H^1(\Omega_h)-$norm, the higher regularity assumption on the differential equation will be needed.

On the other hand, Bornemann and Deuflhard [2] have recently proposed the so-called cascadic multigrid method. As a distinctive feature this method performs more iterations on coarser levels so as to obtain less iterations on finer level. A first candidate of such a cascadic multigrid method was the recently suggested cascadic conjugate gradient method in Deuflhard [7], in short CCG method, which used the CG method as a basic iteration method on each level. The first publication of this algorithm in Deuflhard [7] contained rather convincing numerical results, but no theoretical justification. For the second-order elliptic problem in 2D which is discretized by the P1 conforming element, Bornemann and Deuflhard [2] have proved that the CCG method is accurate with optimal computational complexity for the conjugate gradient method as a smoother and only nearly optimal complexity for other conventional iterative smoother, like symmetric Gauss-Seidel, SSOR or damped Jacobi method. Shi and Xu [11] establish the general framework to analyse the cascadic multigrid method. In [12], Shi and Xu develop the cascadic multigrid method for parabolic problems and obtain the optimal convergence accuracy and computational complexity. Shaidurov and Tobiska [10] study the convergence of the CCG method which is used to solve the elliptic problems in domain with re-entrant corners.

In this paper we use the cascadic multigrid method to solve the second-order elliptic problems with curved boundary discretized by the isoparametric finite element method taking into account numerical integration. We show that in this case the CCG method is accurate with optimal complexity and traditional multigrid smoother (like symmetric Gauss-Seidel, SSOR or damped Jacobi iteration) is accurate with suboptimal complexity.

Apart from the introduction, this paper comprises three sections. In Sect.2, we give a special discretization using the isoparametric finite element. In Sect.3, we take into account into numerical integration and derive the error estimate in L^2–norm over the domain Ω. Finally, we prove the accuracy and complexity of cascadic multigrid method for H^2–regular elliptic problems in Sect.4.

2. A special Discreziation

We start with a coarse approximate triagulation $\mathcal{T}_0$ of Ω for a sufficiently small h_0. The triangulation $\{\tau_k\}$ will be defined from $\{\tau_{k-1}\}$ as follows:

(1). If τ_{k-1} is a triangle with two vertices in Ω then τ_{k-1} is broken into four finer-grid triangles by the line connecting the midpoints of the edges of the triangle τ_{k-1}.

(2). If τ_{k-1} is a triangle with two vertices on $\partial\Omega$, we take a new boundary point to be the crosspoint of the boundary arc and the vertical bisector between the two boundary vertices of τ_{k-1} and form two curved triangles and two straight triangles by connecting the nodes of the element τ_{k-1}. From the construction of $\mathcal{T}_k$, we know that $h_k \approx \frac{1}{2}h_{k-1}$.

In this paper, as in Li [8], we give a special triangulation $\mathcal{T}_k$, that is

$$\overline{\Omega} = \cup_{K\in\mathcal{T}_k}\overline{K}, \quad h_{K,k} = \mathrm{diam}(K), \quad h_k = \max_{K\in\mathcal{T}_k} h_{K,k}, \tag{2.1}$$

where the interior finite element $(K, P_K, \sum_K)(K \in \mathcal{T}_k)$ is obtained from a reference finite element $(\widehat{K}, \widehat{P}, \widehat{\sum})$ through an affine mapping $F_K(\widehat{K})$ which is uniquely determined by the data of the nodes of the finite element K (see [5] and [6]). While the boundary finite element