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A TWO-LEVEL FINITE ELEMENT GALERKIN METHOD FOR THE NONSTATIONARY NAVIER-STOKES EQUATIONS I: SPATIAL DISCRETIZATION *1)

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Abstract

In this article we consider a two-level finite element Galerkin method using mixed finite elements for the two-dimensional nonstationary incompressible Navier-Stokes equations. The method yields a H^1-optimal velocity approximation and a L^2-optimal pressure approximation. The two-level finite element Galerkin method involves solving one small, nonlinear Navier-Stokes problem on the coarse mesh with mesh size H, one linear Stokes problem on the fine mesh with mesh size $h << H$. The algorithm we study produces an approximate solution with the optimal, asymptotic in h, accuracy.

Key words: Navier-Stokes equations, Mixed finite element, Error estimate, Finite element method.

1. Introduction

Two-level finite element Galerkin method is an efficient numerical method for solving nonlinear partial differential equations, e.g., see Xu [24, 25] for steady semi-linear elliptic equations, Layton [14], Ervin, Layton and Maubach [5], Layton and Lenferink [15] and Layton and Tobiska [16] for the steady Navier-Stokes equations. This method is closely related to the nonlinear Galerkin method [1,10, 17-19, 22] and recently developed in [7,21] to solve the nonstationary Navier-Stokes equations. However,it is well known [1, 10, 17-19] that a defect of the nonlinear Galerkin methods is needed to approximate solution u_h as the large eddy component y^H and the small eddy component z^h and solve the unknown components y^H and z^h simultaneously, that is to solve a coupled nonlinear and linear equations and increase computing price.

In the case of the nonlinear evolution problem, the basic idea of the two-level method is to find an approximation u_H by solving a nonlinear problem on a coarse grid with grid size H and find an approximation u^h by solving a linearized problem about the known approximation u_H on a fine grid with grid size h. The semi-discretization in space of the 3D time-dependent Navier-Stokes problem by the two-level method is considered in [7]. Furthermore, the fully discretization in space-time of the 2D and 3D time-dependent Navier-Stokes problem by the two-level method is analyzed in [21], where the local error estimates, stability and convergence are proved, but the global error estimates do not provided. In fact, this scheme is of the global first-order accurate with respect to the time step size τ.

In this report we consider continuity the two-level method used in [21] for the nonstationary, incompressible Navier-Stokes equations and give the error estimates of optimal order for the approximate velocity and pressure. If the equations is discreted by the standard finite element Galerkin method, there will be a large system of nonlinear algebraic equations to be solved.

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To overcome this difficult, we will apply a two-level finite element Galerkin method for solving the nonstationary Navier-Stokes equations in the framework of mixed finite elements. This will yield a small system of nonlinear algebraic equations and a large system of linear algebraic to be solved, i.e., this method can save some computational work. For the standard finite element Galerkin method, the discrete velocity $u_h(\cdot,t)$ and pressure $p_h(\cdot,t)$ are determined in finite element spaces denoted respectively by X_h and M_h which satisfy the so-called inf-sup condition (see [3,8,11]). Our two-level finite element Galerkin method consists in

- Finding $(u_H,p_H)\in(X_H,M_H)$ by solving the nonlinear Navier-Stokes problem on the coarse mesh with width H;
- Finding $(u^h,p^h)\in(X_h,M_h)$ by solving the linear Stokes problem based on (u_H,p_H) on the fine mesh with width $h<<H$.

In this paper, our main results are the following results:

$$\|u^h(t)-u_h(t)\|_{H^1}\le\kappa(t)H^2\quad\forall t\ge 0,\tag{1.1}$$

$$\|p^h(t)-p_h(t)\|_{L^2}\le\kappa(t)H^2\quad\forall t>0\,,\tag{1.2}$$

where (u_h,p_h) is the standard finite element Galerkin approximation based on (X_h,M_h) which satisfies the following error estimates:

$$\|u(t)-u_h(t)\|_{H^1}\le\kappa(t)h,\forall t\ge 0,\tag{1.3}$$

$$\|p(t)-p_h(t)\|_{L^2}\le\kappa(t)h,\forall t>0.\tag{1.4}$$

These estimates indicate that the two-level finite element Galerkin method gives the same order of approximation as the standard finite element Galerkin method if we choose $H=O(h^{1/2})$. However, in our method, the nonlinearity is only treated on the coarse grid and only the linear problem needs to be solved on the fine grid. Of course, the comparison with the standard finite element Galerkin method, the two-level finite element Galerkin method should be made more precise by studying questions related to time discretization and computational implementation. These will be addressed in the several continuations of this work.

2. The Navier-Stokes Equations

Let Ω be a bounded domain in R^2 assumed to have a Lipschitz-continuous boundary Γ and to satisfy a further condition stated in (2.5) below. We consider the time dependent Navier-Stokes equations describing the flow of a viscous incompressible fluid confined in Ω:

$$\frac{\partial u}{\partial t}-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f\quad\text{in }\Omega\,,\,t>0\,,\tag{2.1}$$

$$\text{div}u=0\quad\text{in }\Omega\,,\,t>0\,,\tag{2.2}$$

$$u=0\quad\text{on }\Gamma\,,\,t>0\,,\tag{2.3}$$

$$u(0)=\bar{u}_0\quad\text{in }\Omega\,,\tag{2.4}$$

where $u=(u_1,u_2)$ is the velocity, p is the pressure, f represents the density of body forces, $\nu>0$ is the viscosity and $\bar{u}_0$ is the initial velocity.

In order to introduce a variational formulation, we set

$$X=H_0^1(\Omega)^2\,,\,Y=L^2(\Omega)^2\,,$$

and

$$M=L_0^2(\Omega)=\{q\in L^2(\Omega)\,;\,\int_\Omega q(x)dx=0\}\,.$$