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A TWO-LEVEL FINITE ELEMENT GALERKIN METHOD FOR THE NONSTATIONARY NAVIER-STOKES EQUATIONS II: TIME DISCRETIZATION *1)

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Abstract

In this article we consider the fully discrete two-level finite element Galerkin method for the two-dimensional nonstationary incompressible Navier-Stokes equations. This method consists in dealing with the fully discrete nonlinear Navier-Stokes problem on a coarse mesh with width H and the fully discrete linear generalized Stokes problem on a fine mesh with width $h << H$. Our results show that if we choose $H = O(h^{1/2})$ this method is as the same stability and convergence as the fully discrete standard finite element Galerkin method which needs dealing with the fully discrete nonlinear Navier-Stokes problem on a fine mesh with width h. However, our method is cheaper than the standard fully discrete finite element Galerkin method.

Key words: Navier-Stokes equations, Galerkin method, Finite element.

1. Introduction

Two-level finite element Galerkin method is an efficient numerical method for solving nonlinear partial differential equations, e.g., see Xu [25, 26] for steady semi-linear elliptic equations, Layton [15], Ervin, Layton and Maubach [5], Layton and Lenferink [16] and Layton and Tobiska [17] for the steady Navier-Stokes equations. This method is closely related to the nonlinear Galerkin method [1,11, 18-20, 23] and recently developed in [7,22] to solve the nonstationary Navier-Stokes equations. However,it is well known [1, 11, 18-20, 23] that a defect of the nonlinear Galerkin methods is needed to approximate solution u_h as the large eddy component y^H and the small eddy component z^h and solve the unknown components y^H and z^h simultaneously, that is to solve a coupled nonlinear and linear equations and increase computing price.

In the case of the nonlinear evolution problem, the basic idea of the two-level method is to find an approximation u_H by solving a nonlinear problem on a coarse grid with grid size H and find an approximation u^h by solving a linearized problem about the known approximation u_H on a fine grid with grid size h. The semi-discretization in space of the 3D time-dependent Navier-Stokes problem by the two-level method is considered in [7]. Furthermore, the fully discretization in space-time of the 2D and 3D time-dependent Navier-Stokes problem by the two-level method is analyzed in [22], where the local error estimates, stability and convergence are proved, but the global error estimates do not provided. In fact, this scheme is of the global first-order accurate with respect to the time step size τ.

In the recent work [10] we considered this two-level method used in [22] for solving the nonstationary, incompressible Navier-Stokes equations. If the equations is discreted by the standard finite element Galerkin method, there will be a large system of nonlinear algebraic

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equations to be solved. To overcome this difficult, we applied a two-level finite element Galerkin method for solving the nonstationary Navier-Stokes equations in the framework of mixed finite elements. This will yield a small system of nonlinear algebraic equations and a large system of linear algebraic to be solved, i.e., this method can save a large amount of computational work. For the standard finite element Galerkin method, the discrete velocity $u_h(\cdot,t)$ and pressure $p_h(\cdot,t)$ are determined in finite element spaces denoted respectively by X_h and M_h which satisfy the so-called inf-sup condition (see [3, 8]). Our two-level finite element Galerkin method consists in

- Finding $(u_H, p_H) \in (X_H, M_H)$ by solving the nonlinear Navier-Stokes problem on the coarse mesh with width H;
- Finding $(u^h, p^h) \in (X_h, M_h)$ by solving the linear generalized Stokes problem based on (u_H, p_H) on the fine mesh with width $h << H$.

In recent work [10], our main results are the following results:

$$\|u^h(t) - u_h(t)\|_{H^1} \le \kappa(t)(h + H^2) \quad \forall t \ge 0\,, \tag{1.1}$$

$$\|p^h(t) - p_h(t)\|_{L^2} \le \sigma(t)^{-1/2}\kappa(t)(h + H^2) \quad \forall t > 0\,, \tag{1.2}$$

where $\sigma(t) = \min\{1, t\}$ and (u_h, p_h) is the standard FE Galerkin approximation based on (X_h, M_h) which satisfies the following error estimates:

$$\|u(t) - u_h(t)\|_{H^1} \le \kappa(t)h, \forall t \ge 0, \tag{1.3}$$

$$\|p(t) - p_h(t)\|_{L^2} \le \sigma(t)^{-1/2}\kappa(t)h, \forall t > 0. \tag{1.4}$$

These estimates indicate that the two-level finite element Galerkin method gives the same order of approximation as the standard finite element Galerkin method if we choose $H = O(h^{1/2})$. However, in our method, the nonlinearity is only treated on the coarse grid and only the linear problem needs to be solved on the fine grid.

This paper continues our analysis of the two-level finite element Galerkin method for the Navier-Stokes equations. Here we study the time discretizations of the two-level finite element Galerkin method and the standard finite element Galerkin method in which time is discreted by the Euler implicit difference scheme. By using several discrete analogs of the Gronwall lemma, we are able to show that if we choose $H = O(h^{1/2})$, the two-level finite element Galerkin approximate solution $(u^h_\Delta(t), p^h_\Delta(t))$ is as stable and convergence as what should be verified by the standard finite element Galerkin approximate solution $(u^\Delta_h(t), p^\Delta_h(t))$, namely the numerical solutions $(u^h_\Delta(t), p^h_\Delta(t))$ and $(u^\Delta_h(t), p^\Delta_h(t))$ satisfy

$$\|u^h_\Delta(t)\|_{H^1}, \|u^\Delta_h(t)\|_{H^1} \le c(\|\bar{u}_0\|_{H^1} + \sup_{t\ge 0}\|f(t)\|_{L^2}), \tag{1.5}$$

$$\|u^h_\Delta(t) - u(t)\|_{H^1} \le \kappa(t)(h + H^2 + \Delta t), \forall t \ge 0, \tag{1.6}$$

$$\Big(\int_0^t \|p^h_\Delta(s) - p(s)\|^2_{L^2} ds\Big)^{1/2} \le \kappa(t)(h + H^2 + \Delta t), \forall t \ge 0, \tag{1.7}$$

$$\|u^\Delta_h(t) - u(t)\|_{H^1} \le \kappa(t)(h + \Delta t), \forall t \ge 0, \tag{1.8}$$

$$\Big(\int_0^t \|p^\Delta_h(s) - p(s)\|^2_{L^2} ds\Big)^{1/2} \le \kappa(t)(h + \Delta t), \forall t \ge 0, \tag{1.9}$$

where (1.6)-(1.9) hold if Δt being small. Here $\kappa(t)$ denotes a generic constant depending on the data $(\Omega, \nu, u_0, f_\infty, t)$ and is continuous with respect to time,

$$f_\infty = \sup_{t\ge 0}\{|f(t)| + |f_t(t)|\}\,,\; f_t = \frac{df}{dt};$$