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HOMOGENIZATION OF INCOMPRESSIBLE EULER EQUATIONS *1)

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Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

In this paper, we perform a nonlinear multiscale analysis for incompressible Euler equations with rapidly oscillating initial data. The initial condition for velocity field is assumed to have two scales. The fast scale velocity component is periodic and is of order one. One of the important questions is how the two-scale velocity structure propagates in time and whether nonlinear interaction will generate more scales dynamically. By using a Lagrangian framework to describe the propagation of small scale solution, we show that the two-scale structure is preserved dynamically. Moreover, we derive a well-posed homogenized equation for the incompressible Euler equations. Preliminary numerical experiments are presented to demonstrate that the homogenized equation captures the correct averaged solution of the incompressible Euler equation.

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1. Introduction

In this paper, we study homogenization of the incompressible Euler equation with highly oscillating initial velocity field. The understanding of scale interactions for 3-D incompressible Euler and Navier-Stokes equations has been a major challenge. For high Reynolds number flows, the degrees of freedom are so high that it is almost impossible to resolve all small scales by direct numerical simulations. Deriving an effective equation for the large scale solution is very useful in engineering applications. The nonlinear and nonlocal nature of the Euler equations makes it difficult to construct a properly-posed multiscale solution. If one does not make the correct assumption in the asymptotic expansion of the multiscale solution, one may not be able to derive a well-posed homogenized equation.

The homogenization of the incompressible Euler equation with oscillating data was first studied by McLaughlin, Papanicolaou and Pironneau (MPP for short) in 1985 [8]. To construct

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a multiscale expansion for the solution of the Euler equation, they made a critical assumption that the oscillation is convected by the mean flow. Using multiscale expansion techniques, MPP obtained a periodic cell problem for the velocity field and the pressure. However, it is not clear whether the resulting cell problem has a solution that is periodic in both the fast space variable $\mathbf{y}$ and the fast time variable τ. Even if such solution exists, it may not be unique. Additional assumptions were imposed on the solution of the cell problem in order to derive a variant of the $k-\epsilon$ model.

Our study shows that the small scale oscillations are actually convected by the full oscillatory velocity field. The multiscale structure of the solution becomes apparent when we formulate the Euler equations in vorticity-stream function formulation and use the Lagrangian flow map to describe the propagation of oscillations. Vorticity preserves naturally the multiscale structure of its initial data via the Lagrangian formulation. Velocity can be constructed using the vorticity-stream function formulation. By using a Lagrangian description, we characterize the nonlinear convection of small scales exactly and turn a convection dominated transport problem into an elliptic problem for the stream function. Thus, traditional homogenization result for elliptic problems [1] can be used to obtain multiscale expansions for the stream function and the flow map respectively. Using the insight we obtain from the homogenization theory in the Lagrangian frame, we also derive the corresponding homogenized equation in the Eulerian frame, which can be used more effectively for computational purpose. The effect of viscosity can be included in our analysis. For the sake of simplicity in our presentation, we will not include the viscous effect in the present analysis.

The homogenization theory of the incompressible Euler equation provides a useful guideline in designing effective multiscale computational methods for incompressible flows. In particular, we can use the homogenization theory to examine the role of the so-called Reynolds stress [10, 4]. By making appropriate assumption on the homogeneity of the flow, we can propose new coarse grid model for the large scale solution which is dynamically coupled to a subgrid cell problem. Our multiscale analysis also reveals that without external forcing and viscosity effect, we need to remove certain resonant velocity component in the cell velocity field in order to avoid some secular growth in the multiscale expansion. This resonant cell velocity field corresponds to the non-mixable part of the velocity field. Removing this non-mixable velocity component is equivalent to adding a high frequency forcing term to the incompressible Euler equation, which has the effect of accelerating the flow mixing.

We have performed some preliminary numerical experiments to confirm the convergence of our multiscale analysis. For practical purpose, it is important to generalize our two-scale analysis to problems that do not have scale separation. We can do this either in the physical space or in the Fourier space. From the preliminary computations we present in this paper, we found that the numerical solution obtained from the homogenized equation gives an accurate approximation to the corresponding well-resolved solution of the incompressible Euler equation. Small scale velocity field can be reconstructed from the large scale solution and the subgrid cell solution. Furthermore, we found that there is no need to use a projection method to remove the resonant velocity component if the initial velocity field does not contain such resonant component.

The organization of the rest of the paper is as follows. In Section 2, we will present the formulation of the Euler equations with rapidly oscillating initial data. We will also review the previous work by MPP in some details. Section 3 is devoted to developing the multiscale analysis of the Euler equations in the Lagrangian formulation. We will use the insight gained in Section 3 to perform a multiscale analysis in the Eulerian formulation in Section 4. In Section 5, we will present some numerical examples to demonstrate the convergence of our multiscale analysis.