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LOCAL DISCONTINUOUS GALERKIN METHODS FOR THREE CLASSES OF NONLINEAR WAVE EQUATIONS *1)

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Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

In this paper, we further develop the local discontinuous Galerkin method to solve three classes of nonlinear wave equations formulated by the general KdV-Burgers type equations, the general fifth-order KdV type equations and the fully nonlinear $K(n,n,n)$ equations, and prove their stability for these general classes of nonlinear equations. The schemes we present extend the previous work of Yan and Shu [30, 31] and of Levy, Shu and Yan [24] on local discontinuous Galerkin method solving partial differential equations with higher spatial derivatives. Numerical examples for nonlinear problems are shown to illustrate the accuracy and capability of the methods. The numerical experiments include stationary solitons, soliton interactions and oscillatory solitary wave solutions. The numerical experiments also include the compacton solutions of a generalized fifth-order KdV equation in which the highest order derivative term is nonlinear and the fully nonlinear $K(n,n,n)$ equations.

Mathematics subject classification: 65M60, 35Q53
Key words: Local discontinuous Galerkin method, KdV-Burgers equation, Fifth-order KdV equation, Stability.

1. Introduction

In this paper we further develop the local discontinuous Galerkin method to solve three classes of generalized nonlinear wave equations formulated by the KdV-Burgers type (KdVB) equations

$$u_t + f(u)_x - (a(u)u_x)_x + (r'(u)g(r(u)_x)_x)_x = 0, \tag{1.1}$$

the fifth-order KdV type equations

$$u_t + f(u)_x + (r'(u)g(r(u)_x)_x)_x + (s'(u)h(s(u)_{xx})_{xx})_x = 0, \tag{1.2}$$

and the fifth-order fully nonlinear $K(n,n,n)$ equations

$$u_t + (u^n)_x + (u^n)_{xxx} + (u^n)_{xxxxx} = 0 \tag{1.3}$$

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where $f(u)$, $a(u) \geq 0$, $r(u)$, $s(u)$, $g(p)$ and $h(q)$ are arbitrary (smooth) nonlinear functions. The schemes we present extend the previous work of Yan and Shu [30, 31] and of Levy, Shu and Yan [24] on local discontinuous Galerkin method solving partial differential equations with higher spatial derivatives.

A special case of equation (1.1) is the KdV-Burgers equation

$$u_t + \varepsilon u u_x - \alpha u_{xx} + \beta u_{xxx} = 0 \tag{1.4}$$

derived by Su and Gardner [28], which is a model for nonlinear wave motion incorporating several important physical phenomena, namely dispersion, nonlinear advection and viscosity. The equation arises in the description of long wave propagation in shallow water [2] and in weakly nonlinear plasma physics with dissipation [18]. Efforts to shed light on this problem by means of numerical experiments are made in [3, 6, 33] and the references therein.

The fifth order nonlinear evolution equation

$$u_t + u u_x + u_{xxx} - \delta u_{xxxxx} = 0, \tag{1.5}$$

which is a special case of (1.2), is known as the critical surface-tension model [19]. This equation arises in the modeling of weakly nonlinear waves in a wide variety of media, including magneto-acoustic waves in plasma [23] and long waves in the liquids under ice sheets [25]. There are only a few numerical works in the literature to solve the fifth-order KdV equation. A general type of "semi-localized" solitary wave solutions has been investigated by Kawahara [23], which gave the first numerical evidence of oscillatory solitary wave solutions. Numerical experiments on the semi-localized solutions and their interactions appeared in [4, 5]. In recent numerical studies of break-up of initial data, Hyman and Rosenau [21] have observed a variety of localized pulsating "multiplet" solutions.

The fifth-order fully nonlinear $K(n, n, n)$ equations (1.3) are useful for describing the dynamics of various physical systems. Such nonlinearly dispersive partial differential equations support compacton solutions. A variety of explicit compact solitary wave structures of these fifth-order nonlinear dispersive equations are constructed in [16, 26]. The numerical simulations of these equations in [26] have also revealed the existence of compact traveling breathers. Recently, some attempts have been made to numerically study the stability of the compacton solutions of the fifth-order nonlinear dispersive equations in [15]. The lack of smoothness at the edge of compacton introduces high-frequency dispersive errors into the calculation. It is a challenge to design stable and accurate numerical schemes for solving equation (1.3).

The discontinuous Galerkin method we discuss in this paper is a class of finite element methods using completely discontinuous piecewise polynomial space for the numerical solution and the test functions in the spatial variables, coupled with explicit and nonlinearly stable high order Runge-Kutta time discretization [27]. It was first developed for hyperbolic conservation laws containing first derivatives by Cockburn et al. in a series of papers [11, 10, 8, 12]. For a detailed description of the method as well as its implementation and applications, we refer the readers to the lecture notes [7], the survey paper [9], other papers in that Springer volume, and the review paper [14].

These discontinuous Galerkin methods were generalized to solve a convection diffusion equation (containing second derivatives) by Cockburn and Shu [13]. Their work was motivated by the successful numerical experiments of Bassi and Rebay [1] for the compressible Navier-Stokes equations. Later, Yan and Shu developed a local discontinuous Galerkin method for a general KdV type equation containing third derivatives in [30] and generalized the local discontinuous Galerkin method to PDEs with fourth and fifth spatial derivatives in [31]. Recently, Levy, Shu and Yan [24] developed local discontinuous Galerkin methods for solving nonlinear dispersive equations that have compactly supported traveling wave solutions, the so-called "compactons".

The schemes we present in this paper extend the work of Yan and Shu [30, 31] and of Levy, Shu and Yan [24]. The paper is organized as follows. In section 2 we present local discontinuous