Journal of Computational Mathematics, Vol.22, No.3, 2004, 381–388.

ASYMPTOTIC STABILITY PROPERTIES OF $\theta - METHODS$ FOR THE MULTI-PANTOGRAPH DELAY DIFFERENTIAL EQUATION *1)

Dong-song Li   Ming-zhu Liu
(*Department of mathematics, Harbin Institute of Technology, Harbin 150001, China*)

Abstract

This paper deals with the asymptotic stability analysis of $\theta - methods$ for multi-pantograph delay differential equation

$$\begin{cases} u^{'}(t) = \lambda u(t) + \sum\limits_{i=1}^{l} \mu_i u(q_i t), \quad 0 < q_l < q_{l-1} < \cdots < q_1 < 1, \\ u(0) = u_0. \end{cases}$$

Here $\lambda, \mu_1, \mu_2, \cdots, \mu_l, u_0 \in C$.

In recent years stability properties of numerical methods for this kind of equation has been studied by numerous authors. Many papers are concerned with meshes with fixed stepsize. In general the developed techniques give rise to non-ordinary recurrence relation. In this work, instead, we study constrained variable stpesize schemes, suggested by theoretical and computational reasons, which lead to a non-stationary difference equation. A general theorem is presented which can be used to obtain the characterization of the stability regions of $\theta - methods$.

Mathematics subject classification: 65H10.
Key words: $\theta - methods$, Asymptotic stability, Multi-pantograph delay differential equation.

1. Introduction

Delay differential equations (DDEs) have a wide range of application in applied sciences. Recent studies in diverse fields biology, economy, control and electrodynamics (see for examples[1, 11]) have shown that DDEs play an important role in explaining many different phenomena. In particular they turn out to be fundamental when ODEs-based model fail. DDEs have been studied by many authors who have investigated both their analytical and numerical aspects [2][4][8][12].

The general functional differential equation is given by

$$u'(t) = f(t, u(t), u(\alpha_1(t)), u(\alpha_2(t)), \cdots, u(\alpha_l(t))).$$

A classical case that is the subject of a lot of papers is the following:

$$\alpha_i(t) = t - \tau_i, i = 1, 2, \ldots,$$

where τ_i is a positive constant [6] [7][10]. Another interesting case which is far different from the previous is that the pantograph equation:

$$\begin{cases} u'(t) = f(t, u(t), u(q_1 t), u(q_2 t), \cdots, u(q_l t)), t > 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

* Received December 25, 2001; Final revised December 12, 2003.
1) This work is supported by the NSF of P. R. of China (10271036).

where f is a given function and $0 < q_l < q_{l-1} < \cdots < q_1 < 1$, whereas $u(t)$ is unknown for $t > 0$.

There are many applications for (1.1), for instance, in number theory, in electrodynamics and in the collection of current by the pantograph of an electric locomotive, in nonlinear dynamical systems [5][11].

From a numerical point of view, it is important to study the potential of numerical methods in preserving the qualitative behavior of the analytical solutions. In paper [3] A. bellen and N. Guglielmi investigate the stability properties of $\theta - method$ when it is applied to the following pantograph test equation:

$$\begin{cases} u'(t) = \lambda u(t) + \mu u(qt), t > 0, \\ u(0) = u_0, \end{cases} \tag{1.2}$$

where $\lambda, \mu, u_0 \in C$ and $0 < q < 1$.

In this paper, we study the stability properties of $\theta - methods$ when they are applied to the multi-pantograph test equation:

$$\begin{cases} u'(t) = \lambda u(t) + \sum_{i=1}^{l} \mu_i u(q_i t), \quad 0 < q_l < q_{l-1} < \cdots < q_1 < 1, \quad t > 0, \\ u(0) = u_0. \end{cases} \tag{1.3}$$

Here $\lambda, \mu_1, \mu_2, \cdots, \mu_l, u_0 \in C$.

In section 2 we provided the discretization scheme by applying $\theta - methods$, whose stepsize increase geometrically, to the pantograph equation (1.3).

In section 3 we recall the results concerning the asymptotic stability for the analytical solution of (1.3) and introduce the numerical stability framework. We present the results concerning the stability analysis of $\theta - methods$.

In section 4 we give some numerical experiments to show the asymptotic stability and convergencethe of $\theta - methods$.

2. θ-methods

A. Bellen, N. Guglielmi and L. Torelli described in detail the discretization scheme and constrained global mesh in [3]. We quote their description in the present paper.

Since we are interested in the asymptotic behavior of numerical solution of Eq.(1.3), we suppose to have the numerical solution available till the point $T_0 > 0$.

Firstly we build a primary mesh based on the following relation:

$$T_k = \frac{1}{q_1} T_{k-1}, k = 1, 2, \ldots .$$

In this way we define the primary intervals

$$H_k := T_k - T_{k-1} = \frac{1 - q_1}{q_1^k} T_0, k = 1, 2, \ldots . \tag{2.1}$$

Observe that the sequence increases exponentially. So we define the global mesh H by partitioning every primary interval into a fixed number m of subintervals of the same size. We set

$$h_{n+1} = \frac{H_{[n/m]+1}}{m} = \frac{1}{m} \frac{1 - q_1}{q_1^{[n/m]+1}}, n = 0, 1, 2, \ldots . \tag{2.2}$$

Here $[n/m]$ denotes integer part of n/m.

From (2.2) we have that

$$q_1 h_n = h_{n-m}, n > m.$$

Here for simplicity (but without any loss of generality), we have assumed $t_0 = T_0 = 1$. With $k = n \ mod \ m$, we are now in a position to define the grid points of constrained global mesh H,

$$t_n := T_{[n/m]} + k h_n, n = 1, 2, \ldots . \tag{2.3}$$